Asymptotic entanglement capacity of the Ising and anisotropic Heisenberg interactions

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Outline

- Entanglement as a resource
- Capacities of interactions to produce entanglement
- Two-qubit Hamiltonians: the canonical form
- Capacity of $\mu_x X \otimes X + \mu_y Y \otimes Y$
- Numerical results
- Open problems
Resources in (quantum) information theory

Information is a resource.

- Physical
- Fungible

Examples for two-party problems:

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<th>Dynamic</th>
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<td>cbits_B→A</td>
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<table>
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<th>Quantum</th>
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Quantum information theory is about the interconversion of informational resources.
What is entanglement?

Entangled pure state:

$$\left| \psi \right\rangle_{AB} \neq \left| \phi \right\rangle_{A} \left| \eta \right\rangle_{B}$$

Canonical example: EPR pair

$$\left| \Psi^{+} \right\rangle = \frac{1}{\sqrt{2}} (\left| 0 \right\rangle_{A} \left| 0 \right\rangle_{B} + \left| 1 \right\rangle_{A} \left| 1 \right\rangle_{B})$$

Entanglement = non-classical correlations

- Violation of Bell inequalities
- Can be used to perform classically impossible tasks!
The many uses of entanglement

- Superdense coding [Bennett, Wiesner 92]
- Quantum teleportation [Bennett et al. 93]
- Quantum key distribution [Lo, Chau 98]
- Entanglement-assisted classical communication
  ... through unidirectional channels [Shor et al. 99]
  ... through bidirectional channels [Bennett et al. 02]
- Remote state preparation [Lo 00, Bennett et al. 00]
- Data hiding [DiVincenzo et al. 00]
- Quantum Vernam cipher [Leung 00]
Quantifying entanglement

Consider a bipartite state $|\psi\rangle$.

Any such state has a Schmidt decomposition:

$$|\psi\rangle = \sum_j \sqrt{p_j} |j\rangle_A |\tilde{j}\rangle_B$$

where $\sum_j p_j = 1$ and $\{|j\rangle_A\}$, $\{|\tilde{j}\rangle_B\}$ are orthonormal bases.

Entanglement:

$$E(|\psi\rangle) = -\sum_j p_j \log p_j$$

measured in ebits.

$$1 \text{ ebit} = E(|\Psi^+\rangle)$$
Entanglement is fungible

**Theorem.** Asymptotically, states with the same entanglement are interconvertible.

[| Bennett et al. 95 |

Entanglement concentration

\[ n \text{ copies of } |\psi\rangle \xrightarrow{\text{LO}} nE(|\psi\rangle) \text{ ebits} \]

Entanglement dilution

\[ nE(|\psi\rangle) \text{ ebits} \xrightarrow{\text{LOCC}} n \text{ copies of } |\psi\rangle \]
Physical systems for entanglement generation

- Adjacent quantum dots

- Distant labs connected by optical fiber

General model:

```
A'  \\
A   |
  O
B   |
B'  
```
How to make entanglement

\[ |\psi\rangle \begin{array}{c}
    \{ \\
    A' \\
    A \\
    B \\
    B' \\
\end{array} \begin{array}{c}
    U \\
\end{array} \begin{array}{c}
    \{ \\
    \} \\
\end{array} \nonumber \]

Choose \( |\psi\rangle \) so that \( U|\psi\rangle \) is more entangled than \( |\psi\rangle \).
Entanglement generating capacity

\[ E_U = \sup_{|\psi\rangle \in AA'BB'} \left[ E(U|\psi\rangle) - E(|\psi\rangle) \right] \]

Three technical points:

- Ancillary systems
- Mixed states
- Asymptotic vs. one-shot capacity
Using ancillas

Consider $U = \text{SWAP}$:

$$U|\alpha\rangle|\beta\rangle = |\beta\rangle|\alpha\rangle$$

Clearly $E(|\psi\rangle_{AB}) = E(U|\psi\rangle_{AB})$.

But:

In general, you can make more entanglement when ancillary systems are available. This makes it hard to compute $E_U$!
Mixed states

**Theorem.** For unitary interactions, the optimal input state is always pure.

[Bennett, Harrow, Leung, Smolin 02]

**Proof:**

\[
E'_U = \sup_\rho [D(U \rho U^\dagger) - E_c(\rho)] \\
\leq \sup_\rho [E_c(U \rho U^\dagger) - E_c(\rho)] \\
= \frac{1}{n} \sup_\rho \left[ E_f((U \rho U^\dagger)^\otimes_n) - E_f(\rho^\otimes_n) \right] + \epsilon \\
= \frac{1}{n} \sum_i p_i \left[ E((U|\psi_i\rangle)^\otimes_n) - E(|\psi_i\rangle^\otimes_n) \right] + \epsilon \\
= \sum_i p_i \left[ E(U|\psi_i\rangle) - E(|\psi_i\rangle) \right] \\
= \sup_\rho \sup_{i} \left[ E(U|\psi_i\rangle) - E(|\psi_i\rangle) \right] \\
= E_U
\]
Asymptotic vs. one-shot

**Theorem.** $E^{(n)}_U = nE_U$

[Bennett, Harrow, Leung, Smolin 02]

**Proof:**

The entanglement can only increase by application of $U$. For each use of $U$, the maximum increase is given by $E_U$. Thus $E^{(n)}_U \leq nE_U$.

By using the optimal input $n$ times, $E^{(n)}_U \geq nE_U$. □
Entanglement production cycle

Create initial entanglement (inefficiently)

Dilute $nE(|\psi\rangle)$ ebits
\[ \downarrow \]
\[ |\psi\rangle^{\otimes n} \]

Excess entanglement: $nE_U$ ebits

Apply $U^{\otimes n}$

Concentrate $(U|\psi\rangle)^{\otimes n}$
\[ \downarrow \]
\[ nE(U|\psi\rangle) \text{ ebits} \]
Entanglement capacity of a Hamiltonian

\[ E_H = \lim_{t \to 0} (E e^{-iHt} / t) \]

\[ = \sup_{|\psi\rangle} \left[ \frac{d}{dt} E(e^{-iHt}|\psi\rangle) \right]_{t=0} \]

Using perturbation theory, we find

\[ E_{H,|\psi\rangle} = \sum_{j,k} \sqrt{p_j p_k} \log(p_j / p_k) \, \text{Im} \langle j \tilde{j} | H | k \tilde{k} \rangle \]

where

\[ |\psi\rangle = \sum_{j} \sqrt{p_j} |j\rangle_{AA'} |j\rangle_{BB'} \]

This is...

- Zero for product states
- Zero for maximally entangled states
- Hard to optimize over $|\psi\rangle$!
Two-qubit Hamiltonians: Canonical form

A general two-qubit Hamiltonian has 16 real parameters. But only two of them are nonlocal!

Fact: Any two-qubit Hamiltonian $H$ is locally equivalent to a Hamiltonian of the form

$$\tilde{H} = \mu_x X \otimes X + \mu_y Y \otimes Y + \mu_z Z \otimes Z.$$ 

In other words, there are local Hamiltonians $H_A, H_B$ and local unitary operators $U, V$ so that

$$H = (U \otimes V) \tilde{H} (U^\dagger \otimes V^\dagger) + H_A + H_B.$$ 

[Dür et al. 01]

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
**Ising interaction**

Consider $H = Z \otimes Z$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that $H$ is locally equivalent to $2\ket{00}\bra{00}$.

No ancillas:

$$E^*_{ZZ} = 4 \max_{p,|\psi\rangle} \sqrt{p(1-p)} \log \frac{p}{1-p} \text{Im}(\langle \psi |00\rangle \langle 00| \psi^\perp \rangle)$$

$$= 2 \max_p \sqrt{p(1-p)} \log \frac{p}{1-p}$$

$$\approx 1.9123$$

[Dür et al. 01]

**Theorem.** $E_{ZZ} = 1.9123$

[Childs, Leung, Vidal, Verstraete 02]

**Proof idea:** No pair of terms in the Schmidt decomposition with Schmidt coefficients $p_1, p_2$ can contribute more than $E^*_{ZZ}/(p_1 + p_2)$.
\[ \mu_x \mathbf{XX} + \mu_y \mathbf{YY} \]

**Upper bound:** Simulation.

The Hamiltonian \( \mu_x \mathbf{X} \otimes \mathbf{X} + \mu_y \mathbf{Y} \otimes \mathbf{Y} \) can be *simulated* using \((\mu_x + \mu_y) \mathbf{Z} \otimes \mathbf{Z}\).

- There exist unitaries \( H, K \) so that
  \[ HZH^\dagger = X \quad KZK^\dagger = Y \]

- Use the Lie product formula
  \[ e^{-i(H_1+H_2)t} = \lim_{n \to \infty} (e^{-iH_1t/n} e^{-iH_2t/n})^n \]

Therefore \( E_{\mu_x \mathbf{XX} + \mu_y \mathbf{YY}} \leq (\mu_x + \mu_y) E_{\mathbf{ZZ}} \).

**Lower bound:** By the explicit protocol (with no ancillas), \( E_{\mu_x \mathbf{XX} + \mu_y \mathbf{YY}} \geq (\mu_x + \mu_y) E_{\mathbf{ZZ}} \). \[\text{[Dür et al. 01]}\]
Summary of known capacities

Gates:
\[ E_{\text{CNOT}} = 1 \]
\[ E_{\text{SWAP}} = 2 \]

Hamiltonians:
\[ E_{\mu_x XX + \mu_y YY} = 1.9123(\mu_x + \mu_y) \]

In general, there may be no closed form expression for the capacity of a given interaction.

For the Hamiltonian
\[ H = \mu_{xy}(X \otimes X + Y \otimes Y) + Z \otimes Z \]
we conjecture
\[
E_{\mu_{xy}(XX+YY)+ZZ} = 2 \max \left\{ \sqrt{p_1p_2} \log(p_1/p_2) \left[ \sin \theta + \mu_{xy} \sin(\varphi - \xi) \right] + \sqrt{p_2p_4} \log(p_2/p_4) \left[ \sin \varphi + \mu_{xy} \sin(\theta - \xi) \right] + \sqrt{p_1p_4} \log(p_1/p_4) \mu_{xy} \sin \xi \right\}
\]

where \( p_1, p_2, p_4 > 0, p_1 + 2p_2 + p_4 = 1, \) and \( \theta, \varphi, \xi \in [0, 2\pi) \).
Open problems

• Calculate capacities for two-qubit gates

• Find an upper bound on the optimal ancilla dimension for a $d_A \times d_B$ dimensional gate or Hamiltonian

• Study entanglement generation by nonunitary quantum operations

• Inverse problem: How much entanglement is needed to simulate a gate (or Hamiltonian)?

$$E_U \leq \text{ebits needed to simulate } U$$

When is this achievable?