

where α_j is chosen so that each column of F_n has norm one. Then $F_n^T F_n = I$ and $F_n^T T F_n = \Gamma \equiv \text{diag}(\gamma_1, \dots, \gamma_n)$, with $\gamma_j = 4 - 2\beta_j$ and $\beta_j = \cos \frac{j\pi}{n+1}$. Now let

$$\bar{A} = \begin{bmatrix} F_n^T & & & & \\ & F_n^T & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & F_n^T \end{bmatrix} A \begin{bmatrix} F_n & & & & \\ & F_n & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & F_n \end{bmatrix} = \begin{bmatrix} \Gamma & -I & & & \\ -I & \Gamma & -I & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -I & \Gamma \end{bmatrix}.$$

Let P be the permutation matrix that reorders the rows of the identity matrix as $1, n+1, 2n+1, \dots, (m-1)n+1, 2, n+2, 2n+2, \dots, (m-1)n+2$, etc. Then

$$P\bar{A}P^T = \begin{bmatrix} T_1 & & & & \\ & T_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & T_n \end{bmatrix}$$

where

$$T_j = \begin{bmatrix} \gamma_j & -1 & & & \\ -1 & \gamma_j & -1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -1 & \gamma_j \\ & & & & -1 & \gamma_j \end{bmatrix}$$

has dimension $m \times m$. Now we can see that

$$\begin{aligned} & \begin{bmatrix} F_m^T & & & & \\ & F_m^T & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & F_m^T \end{bmatrix} \begin{bmatrix} T_1 & & & & \\ & T_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & T_n \end{bmatrix} \begin{bmatrix} F_m & & & & \\ & F_m & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & F_m \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_1 & & & & \\ & \Gamma_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \Gamma_n \end{bmatrix}, \end{aligned}$$

where $\Gamma_j = \text{diag}(\gamma_j - 2\beta_1, \dots, \gamma_j - 2\beta_m)$.

Putting all this information together, we have the eigensystem of the matrix A :

$$UAU^T = \Lambda$$

with

$$U = \begin{bmatrix} F_m^T & & & & \\ & F_m^T & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & F_m^T \end{bmatrix} P \begin{bmatrix} F_n^T & & & & \\ & F_n^T & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & F_n^T \end{bmatrix}$$

and $\Lambda = \text{diag}(\lambda_{jk})$ with

$$\lambda_{jk} = 4 - 2 \cos \frac{j\pi}{n+1} - 2 \cos \frac{k\pi}{m+1}.$$

Solving Linear Systems $Az = b$

We can now give an efficient algorithm for solving the linear system $Az = b$ where $z = [z_{11}, \dots, z_{1n}, \dots, z_{m1}, \dots, z_{mn}]^T$. It uses the representation $z = U^T \Lambda^{-1} U b$. It takes advantage of the fact that multiplication by the matrix F_p or F_p^T is equivalent to a discrete Fourier transform of length p . This can be accomplished in $O(p \log_2 p)$ operations if p is a power of 2 and in some larger but still modest number of operations if p is a composite number with many factors. For convenience, we'll assume that m and n are powers of 2.

Step 1: Form

$$\hat{b} = \begin{bmatrix} F_n^T & & & & \\ & F_n^T & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & F_n^T \end{bmatrix} b$$

by performing m discrete Fourier transforms

$$\hat{b}_k = F_n^T \begin{bmatrix} b_{k1} \\ \vdots \\ b_{kn} \end{bmatrix}, k = 1, \dots, m.$$

Cost: $O(mn \log_2 n)$.

Step 2: Form

$$c = \begin{bmatrix} F_m^T & & & & \\ & F_m^T & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & F_m^T \end{bmatrix} \hat{b}_{\text{permuted}}$$

by forming

$$c_j = F_m^T \begin{bmatrix} \hat{b}_{1j} \\ \cdot \\ \cdot \\ \hat{b}_{mj} \end{bmatrix}, j = 1, \dots, n.$$

Cost: $O(mn \log_2 m)$.

Step 3: Form $\bar{c} = \Lambda^{-1}Ub$ by computing $\bar{c}_{kj} = c_{kj}/\lambda_{kj}$, $k = 1, \dots, m$, $j = 1, \dots, n$.

Cost: $O(mn)$

Step 4: Form

$$\hat{c} = \begin{bmatrix} F_m & & & & \\ & F_m & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & F_m \end{bmatrix} \bar{c}$$

by forming

$$\hat{c}_j = F_m \begin{bmatrix} \bar{c}_{1j} \\ \cdot \\ \cdot \\ \bar{c}_{mj} \end{bmatrix}, j = 1, \dots, n.$$

Cost: $O(mn \log_2 m)$.

Step 5: Form

$$z = \begin{bmatrix} F_n & & & & \\ & F_n & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & F_n \end{bmatrix} \hat{c}_{permuted}$$

by forming

$$z_k = F_n \begin{bmatrix} \hat{c}_{k1} \\ \cdot \\ \cdot \\ \hat{c}_{kn} \end{bmatrix}, k = 1, \dots, m.$$

Cost: $O(mn \log_2 n)$.

For further information on this method and other fast direct methods for solving this and similar problems, see

Paul N. Swarztrauber, "The Methods of Cyclic Reduction, Fourier Analysis, and the FACR Algorithm for the Discrete Solution of Poisson's Equation on a Rectangle," *SIAM Review* 19 (1977) 490-501.