AMSC/CMSC 661 Scientific Computing II
Spring 2005
Solution of Ordinary Differential Equations, Initial Value Problems

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These notes are based on the 2003 textbook of Stig Larsson and Vidar Thomée.

Ordinary Differential Equations, Initial Value Problems

Ordinary Differential Equations, Initial Value Problems = ODE/IVP

The plan:

- A review of the linear problem and initial conditions
- A few numerical methods

A recurring theme: Stability.

Note: We will assume that any matrix A that we use has a complete set of eigenvalues and eigenvectors. Almost all matrices do. In that case, A can be factored as in an **eigendecomposition** as

$$A = W \Lambda W^{-1}$$

where Λ is a diagonal matrix with entries λ_j, the eigenvalues of A.

The philosophy:

- Some of this material is covered in 660.
- We'll just do what we need to cover IVPs for PDEs.

Reference: Chapter 7.

A review of the linear problem and initial conditions

A single first order ODE

Problem: Find the function $u(t) : \mathcal{R} \to \mathcal{R}$ that satisfies

$$u' + au = f(t)$$

for $t > 0$, with $u(0) = v$, a given number and a a given number.

Jargon: The equation is called **first order** because the highest derivative is the first.

Solution:

$$u(t) = e^{-at}v + \int_0^t e^{-a(t-s)} f(s) ds.$$

Unquiz 1: Verify that this solution satisfies the differential equation and the initial value condition.

A system of first order ODEs

Problem: Find the function $u(t) : \mathcal{R}^1 \to \mathcal{R}^n$ that satisfies

$$u' + Au = f(t)$$

for $t > 0$, with $u(0) = v$, a given vector and A a given $n \times n$ matrix.

Solution:

$$u(t) = e^{-tA}v + \int_0^t e^{-(t-s)A} f(s) ds,$$

where

$$e^B = \sum_{j=0}^{\infty} \frac{1}{j!} B^j.$$

Unquiz 2: Verify that this solution satisfies the differential equation and the initial value condition.

A more useful expression for the matrix exponential function

If $A = W\Lambda W^{-1}$ where Λ is a diagonal matrix with entries λ_j, then

$$\begin{aligned} e^A &= \sum_{j=0}^{\infty} \frac{1}{j!} A^j \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} (W\Lambda W^{-1})^j \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} W\Lambda^j W^{-1} \\ &= W \left(\sum_{j=0}^{\infty} \frac{1}{j!} \Lambda^j \right) W^{-1} \end{aligned}$$

Aside: Taylor series tells us that

$$\begin{aligned} e^\lambda &= e^0 + \lambda e^0 + \frac{1}{2}\lambda^2 e^0 + \frac{1}{3!}\lambda^3 e^0 + \dots \\ &= 1 + \lambda + \frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3 + \dots \\ &= \sum_{j=0}^{\infty} \frac{1}{j!}\lambda^j. \end{aligned}$$

So we conclude that

$$e^A = W \exp(\Lambda) W^{-1}$$

where $\exp(\Lambda)$ is a diagonal matrix with entries e^{λ_j}.

Observation:

- Since the solution to the ODE is

  $$u(t) = e^{-tA}v + \int_0^t e^{-(t-s)A} f(s) ds,$$

  we see that as $t \to \infty$, $u(t) \to 0$ if all eigenvalues of A are positive. This is called **asymptotic stability**. In this case, small changes in the data make small changes in the solution.

- If A has any negative eigenvalues, then the solution u can grow as $t \to \infty$. This is called **instability**.

A single second order ODE

Problem: Find the function $u(t) : \mathcal{R} \to \mathcal{R}$ that satisfies

$$u'' + au = f(t)$$

for $t > 0$, with $u(0) = v, u'(0) = w$, (given numbers) and a a given number.

Solution:

$$u(t) = \cos(t\sqrt{a})v + \frac{1}{\sqrt{a}} \sin(t\sqrt{a})w$$

if $f = 0$.

Unquiz 3: Verify that this solution satisfies the differential equation and the initial value conditions.

A system of second order ODEs

Problem: Find the function $u(t) : \mathcal{R}^1 \to \mathcal{R}^n$ that satisfies

$$u'' + Au = f(t)$$

for $t > 0$, with $u(0) = v, u'(0) = w$, (given vectors) and A a given $n \times n$ matrix.

Solution:

$$u(t) = \cos(t\sqrt{A})v + (\sqrt{A})^{-1}\sin(t\sqrt{A})w$$

if $f = 0$, where

$$\begin{aligned} \cos B &= \frac{1}{2}(e^{iB} + e^{-iB}) \\ \sin B &= \frac{1}{2i}(e^{iB} - e^{-iB}) \\ \sqrt{A} &= W\sqrt{\Lambda}W^{-1} \end{aligned}$$

and $\sqrt{\Lambda}$ has diagonal entries $\sqrt{\lambda_j}$.

Unquiz 4: Verify that this solution satisfies the differential equation and the initial value conditions.

Three numerical methods for first order equations

- Euler
- backward Euler
- Crank-Nicolson

Numerical methods for a single ODE

The single ODE:

$$u' = f(t, u)$$

Let k be the mesh spacing for t (just as h was for the variable x).

- **Euler** (an **explicit method**)

$$\frac{u(t+k) - u(t)}{k} = f(t, u(t)).$$

- **backward Euler** (an **implicit method**)

$$\frac{u(t+k) - u(t)}{k} = f(\mathbf{t+k}, u(t+k)).$$

- **Crank-Nicolson** (an **implicit method**)

$$\frac{u(t+k) - u(t)}{k} = f(t + k/2, (u(t+k) + u(t))/2).$$

(The generalization to a system of equations is clear(???).)

Accuracy of these three methods

– Euler
$$\frac{u(t+k)-u(t)}{k} = f(t,u(t)).$$
Taylor series expansion says we make an error of order k.

– backward Euler
$$\frac{u(t+k)-u(t)}{k} = f(\mathbf{t+k},u(t+k)).$$
Taylor series expansion says we make an error of order k.

– Crank-Nicolson
$$\frac{u(t+k)-u(t)}{k} = f(t+k/2,(u(t+k)+u(t))/2).$$
Taylor series expansion says we make an error of order k^2.

Stability analysis of these three methods

Suppose we apply them to the single ODE
$$u' + au = 0,$$
and let u^n be the approximate solution we obtain for $t = nk$. Then **Euler's method** gives
$$u(t+k) = u(t) - kau(t)$$
so
$$\begin{aligned} u^0 &= v \\ u^1 &= (1-ka)v \\ u^n &= (1-ka)^n v \end{aligned}$$

Unquiz 5: Show that the Backward Euler method gives
$$u^n = \frac{1}{(1+ka)^n}v.[]$$

In a similar way, you could show that Crank-Nicholson gives
$$u^n = \left(\frac{1-ka/2}{1+ka/2}\right)^n v.$$

So what?

Euler:	$u^n = (1 - ka)^n v$
Backward Euler:	$u^n = \frac{1}{(1+ka)^n} v$
Crank-Nicholson:	$u^n = \left(\frac{1-ka/2}{1+ka/2}\right)^n v$
True solution:	$u(nk) = e^{-nka} v$

Suppose $a > 0$, so that the ODE is asymptotically stable. When are our approximations asymptotically stable?

Euler:	$u^n = (1 - ka)^n v$ Stable if $\lvert 1 - ka \rvert < 1$, or $k < 2/a$.
Backward Euler:	$u^n = \frac{1}{(1+ka)^n} v$ Stable unconditionally.
Crank-Nicholson:	$u^n = \left(\frac{1-ka/2}{1+ka/2}\right)^n v$ Stable unconditionally.

A numerical method for a second order equation

$$u'' = f(t, u)$$

becomes

$$\frac{u(t+k) - 2u(t) + u(t-k)}{k^2} = f(t, u(t)).$$

The error is $O(k^2)$, and the method is stable on the linear problem $u'' + au = 0$ for any k and a.