AMSC/CMSC 661 Scientific Computing II
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Solution of Hyperbolic Partial Differential Equations
Part 2: Numerics
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These notes are based on the 2003 textbook
of Stig Larsson and Vidar Thomée.

The plan:

Recall: Hyperbolic equations come in three forms:

- First order scalar equations, analyzed by characteristics. We'll use finite differences.
- Symmetric hyperbolic systems, decoupled by matrix eigenvectors and solved by characteristics. We'll use finite differences.
- The wave equation, analyzed by eigenfunctions of $-\Delta u$. We'll use finite elements in class and finite differences in the homework.

Stability proofs are similar to those for parabolic equations, and we will omit them.

Special challenges from hyperbolic equations

- Discontinuities are preserved, so they must be approximated well.
- Conservation of energy is important and (ideally) should be preserved by the numerical method.

First order scalar equations and characteristics

What we know

$$u_t + a \cdot \bigtriangledown u + a_0 u = f$$

for $x \in \Omega \subset \mathcal{R}^d$, $t \in \mathcal{R}_+$, with initial conditions

$$u(x, 0) = v(x)$$

for $x \in \Omega$, and boundary conditions

$$u(x, t) = g(x, t)$$

for (x, t) on the **inflow boundary**.

Assume that there is no point x for which $a = 0$, and assume that all of the coefficients are smooth.

Define the **characteristic**, or **streamline**, to be the solution $x(s)$ to the system of ordinary differential equations

$$\frac{dx_j(s)}{ds} = a_j(x(s)).$$

The solution along the entire characteristic depends only on the single initial value $v(x_0)$ at the point where the characteristic starts.

Numerical methods

The simplest possible problem is

$$u_t = au_x,$$

with

$$u(x, 0) = v(x).$$

Note that we have switched the sign of a from the previous slide. We could use **finite differences** on both terms:

$$\frac{u_j^{n+1} - u_j^n}{k} = a\frac{u_{j+1}^n - u_j^n}{h},$$

yielding

$$u_j^{n+1} = aru_{j+1}^n + (1 - ar)u_j^n$$

$(r = k/h)$, which (we see) is stable if $1 - ar > 0$.

This finite difference method is 1st order in k and h.

Unquiz: Try this method

$$u_j^{n+1} = aru_{j+1}^n + (1 - ar)u_j^n$$

with $a = 1$ and $v(x)$ a step function that is 1 for $1/2 \leq x \leq 1$ and 0 elsewhere.

Then try it for $a = -1$ and the same $v(x)$.

How well does it work? []

From the 2nd part of the unquiz, we see that it is important to **take the x difference in the upwind direction**.

The general principle is called the **Courant-Friedrichs-Lewy (CFL) condition for stability**: in order for a method to be stable, it is necessary that the domain of dependence for the numerical method contain the domain of dependence for the differential equation.

Higher order methods

Note that we can construct methods of $O(h^p)$ by taking more than 2 points in our approximation to u_x. See pp. 187-189

It would be natural to use a **symmetric difference** instead of an upwind one. Unfortunately, this method is **unstable** for all choices of r unless we also use a more complicated approximation to u_t.

Some useful difference methods

Friedrichs method

$$\frac{u_j^{n+1} - u_j^n}{k} = a\frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2k}$$

Notice that the last term is $h/(2r)$ times an approximation to u_{xx}. This addition of an **artificial diffusion term** (sometimes called **artificial viscosity**) is a common way to stabilize methods. The Friedrichs method is **first order accurate** and requires $-1 \leq ar \leq 1$ for stability.

Lax-Wendroff method

$$u_j^{n+1} = \alpha u_{\mathbf{j+1}}^n + \beta u_j^n + \gamma u_{\mathbf{j-1}}^n$$

with

$$\begin{aligned} \alpha &= \frac{a^2r^2 + ar}{2}, \\ \beta &= 1 - a^2r^2, \\ \gamma &= \frac{a^2r^2 - ar}{2}. \end{aligned}$$

This is stable in the L_2 norm when $a^2r^2 \leq 1$ but not stable in the max-norm. It is 2nd order accurate in space.

Wendroff box method for $u_t + au_x + a_0u = f$

$$\begin{aligned} &\frac{u_j^{n+1} + u_{j+1}^{n+1} - u_j^n - u_{j+1}^n}{2k} \\ &+a\frac{u_{j+1}^{n+1} + u_{j+1}^n - u_j^{n+1} - u_j^n}{2h} \\ &+a_0\frac{u_j^{n+1} + u_{j+1}^{n+1} + u_j^n + u_{j+1}^n}{4} \quad = \quad f_j^n \end{aligned}$$

This is stable in L_2 when $1 - ar > 0$, and 2nd order accurate in space.

Symmetric hyperbolic systems

For $x \in \mathcal{R}$ and $t \geq 0$, let $u, f : \mathcal{R}^2 \rightarrow \mathcal{R}^n$ satisfy

$$u_t + A(x,t)u_x + B(x,t)u = f(x,t)$$

with initial values $u(x,0) = v(x)$.

Suppose A, B, and f are smooth functions.

The methods we discussed for the scalar equation all have generalizations to symmetric hyperbolic systems. See Section 12.2.

The IBVP for the wave equation

Reference: Section 13.1.

The equation:

$$\begin{aligned} u_{tt} - \Delta u(x,t) = &\quad 0 &\quad \text{for } x \in \Omega \subset \mathcal{R}^2, t \in \mathcal{R}_+ \\ u(x,0) = &\quad v(x) &\quad \text{for } x \in \Omega \\ u_t(x,0) = &\quad s(x) &\quad \text{for } x \in \Omega \\ u(x,t) = &\quad 0 &\quad \text{for } x \in \Gamma(\Omega), t \in \mathcal{R}_+ \end{aligned}$$

Assume that the **boundary** of $\Omega \subset \mathcal{R}^2$ is a convex polygon.

Let's use piecewise linear finite elements, expressing

$$\mathbf{u_h}(x,t) = \sum_{i=1}^{M} \alpha_j(t)\phi_j(x)$$

where ϕ_j is a hat function.

In **weak form**, our equation is

$$((u_h)_{tt}, \phi) + a(u_h, \phi) = (f, \phi),$$

$$u_h(x,0) = v_h(x), \;\; (u_h)_t(x,0) = \mathbf{s}_h(x),$$

where v_h and s_h are piecewise linear approximations to v and s:

$$v_h = \sum_{j=1}^{M} \beta_j \phi_j(x), \;\; \mathbf{s_h} = \sum_{j=1}^{M} \gamma_j \phi_j(x),$$

This gives us a system of ordinary differential equations for the coefficients α:

$$\mathbf{B}\alpha''(t) + \mathbf{A}\alpha(t) = \mathbf{f}(t)$$

where

$$\begin{aligned} b_{mj} &= (\phi_m, \phi_j) \\ a_{mj} &= a(\phi_m, \phi_j) \\ f_m &= (f, \phi_m) \end{aligned}$$

and the initial conditions are

$$\alpha(0) = \beta, \ \ \alpha'(0) = \gamma.$$

Energy conservation

Recall that for the wave equation, the energy

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega u_t^2 + | \bigtriangledown u|^2 dx$$

is constant when $f = 0$. Is this true for our discrete problem?

Let's take our weak formulation

$$((u_h)_{tt}, \phi) + a(u_h, \phi) = 0$$

This is true for all piecewise linear functions, so let's use $(u_h)_t$:

$$((u_h)_{tt}, (u_h)_t) + a(u_h, (u_h)_t) = 0.$$

Now notice that this is just

$$\frac{1}{2} \frac{d}{dt} \left(\|(u_h)_t\|^2 + |u_h|_1^2 \right) = 0,$$

so the energy is conserved.

Error formula

Theorem: (13.2, p204) Given a function $g(x)$, Let $R_h g$ be the piecewise linear function that interpolates at the meshpoints. If we take the initial values

$$u^0 = R_h v,$$

$$u^1 = R_h(v + k(s) + \frac{k^2}{2}(\Delta v + f(\cdot, 0))),$$

then

$$\|u^{n+1/2} - u(t_n + k/2)\| + \|(u^{n+1} - u^n)/k - u_t(t_n + k/2)\| \le C(h^2 + k^2),$$

where C depends on u and t_n.

Also,

$$|u^{n+1/2} - u(t_n + k/2)|_1 \le C(h + k^2).$$

Final comments

Solving hyperbolic equations is more difficult (in fact, more of an art) than solving parabolic or elliptic:

- Many obvious difference formulas are unstable.
- You need to know a fair amount about the solution before you begin; for example, characteristic directions, inflow boundaries, domain of dependence.
- The solution does not smooth out with time, so **shock waves** and other discontinuities persist.
- Conservation of energy is an important consideration.

We have just touched the surface of this subject.