AMSC/CMSC 661 Scientific Computing II
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Solution of Hyperbolic Partial Differential Equations
Part 1: Theory

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These notes are based on the 2003 textbook of Stig Larsson and Vidar Thomée.

Recall from Introduction to our Course:

Consider a differential equation that is a function of two variables, x and t:

$$au_{tt} + 2bu_{xt} + cu_{xx} + \ldots = f(x, t)$$

where the dots denote terms that have fewer than 2 derivatives. We classify the differential equation depending on a, b, and c:

- **elliptic** if $ac - b^2 > 0$.
  Example: **Poisson's equation** $u_{tt} + u_{xx} = f(x, t)$.
- **hyperbolic** if $ac - b^2 < 0$.
  Example: the **wave equation** $u_{tt} - u_{xx} = f(x, t)$.
- **parabolic** if $ac - b^2 = 0$.
  Example: the **heat equation** $u_t - u_{xx} = f(x, t)$.

It is now time to study **hyperbolic equations**.

The plan:

First, some of the theory (Chapter 11).
Hyperbolic equations come in three forms:

- The wave equation
- First order scalar equations
- Symmetric hyperbolic systems

What makes all of them different from

- elliptic equations (where the solution at each point is coupled to every other point)
- and parabolic equations (in which the solution now depends on what happens at every point in history)

is their dependence on only a small part of the historical data.

The IBVP for the wave equation

Reference: Section 11.2

The equation:

$$\begin{aligned} \mathbf{u_{tt}} - \Delta u(x,t) &= 0 && \text{for } x \in \Omega \subset \mathcal{R}^d, t \in \mathcal{R}_+ \\ u(x,0) &= v(x) && \text{for } x \in \Omega \\ \mathbf{u_t(x,0)} &= \mathbf{s(x)} && \text{for } x \in \Omega \\ u(x,t) &= 0 && \text{for } x \in \Gamma(\Omega), t \in \mathcal{R}_+ \end{aligned}$$

This looks very much like the Initial-Boundary Value Problem for the heat equation, except for the parts in blue.

The right tool for analyzing the IBVP for the heat equation was the **eigendecomposition**, and we use it here, too.

Let's pull out our complete set of orthonormal eigenfunctions for $-\Delta u = \lambda u$ on Ω.

Call the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and call the eigenfunctions z_j, so that

$$-\Delta z_j = \lambda_j z_j \text{ in } \Omega,$$

and

$$z_j = 0 \text{ on } \Gamma.$$

We'll **separate variables** and try to express u as a sum of z's:

$$u(x,t) = \sum_{j=1}^{\infty} \mathbf{w_j(t)} z_j(x).$$

Let's see if we can find functions $w_j(t)$ to make this work.

Note that the **boundary conditions** are automatically satisfied.

Let's try to **satisfy the PDE**:

$$\begin{aligned} u_{tt} &= \sum_{j=1}^{\infty} w_j''(t) z_j(x) \\ -\Delta u &= \sum_{j=1}^{\infty} w_j(t) \lambda_j z_j(x) \end{aligned}$$

so

$$u_{tt} - \Delta u = \sum_{j=1}^{\infty} (w_j''(t) + \lambda_j w_j(t)) z_j(x) = 0.$$

Since the z_j form an **orthonormal** basis, we must have

$$w_j'' + \lambda_j w_j = 0$$

for $t > 0$, with

$$w_j(0) = (v, z_j),$$
$$w_j'(0) = (s, z_j)$$

and we know the solution to ODEs that look like this:

$$w_j(t) = (v, z_j)\cos(\mathbf{t}\sqrt{\lambda_j}) + \frac{(\mathbf{s}, \mathbf{z_j})}{\sqrt{\lambda_j}} \sin(\mathbf{t}\sqrt{\lambda_j}),$$

so

$$u(x,t) = \sum_{j=1}^{\infty} ((\mathbf{v}, \mathbf{z_j})\cos(\mathbf{t}\sqrt{\lambda_j}) + \frac{(\mathbf{s}, \mathbf{z_j})}{\sqrt{\lambda_j}} \sin(\mathbf{t}\sqrt{\lambda_j})) z_j(x).$$

Therefore, we know that a solution exists as long as this series converges.

Well-posedness

We are one step toward establishing well-posedness. We still need uniqueness and stability, consequences of this theorem:

Theorem: If u is sufficiently smooth, then the **energy of the solution**

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} (u_t(x,t)^2 + |\bigtriangledown u(x,t)|^2) dx$$

is constant with respect to time.

Proof: Theorem 11.2, p167.

Unquiz: Explain how this implies uniqueness and stability.

The cone of influence

Suppose we are interested in the solution to the wave equation at a single point $(\bar{x}, \bar{t})$, where $t > 0$.

Question: How much initial data do we need in order to compute the solution at this point?

The answer is somewhat surprising, given our experience with elliptic and parabolic equations.

Answer: We need the initial data in a d-dimensional sphere of radius $\bar{t}$ around $\bar{x}$.

Proof: See Theorem 11.3, p.167. (but not very intuitive).

First order scalar equations and characteristics

Reference: Section 11.3 is not easy to read. We'll do it by example.

This form of the problem looks like

$$u_t + a \cdot \bigtriangledown u + a_0 u = f$$

for $x \in \Omega \subset \mathcal{R}^d$, $t \in \mathcal{R}_+$, with initial conditions

$$u(x, 0) = v(x)$$

for $x \in \Omega$.

The **boundary conditions** are rather special:

$$u(x, t) = g(x, t)$$

for (x, t) on the **inflow boundary**.

- This boundary is defined by the points $x \in \Gamma(\Omega)$ and $t > 0$ for which $a \cdot n < 0$.
- n, which depends on x, is the **exterior normal**, the unit vector pointing out from Ω, perpendicular to Γ.

We don't specify boundary conditions on the rest of the boundary; if we do, the solution may fail to exist.

Assume that there is no point x for which $a = 0$, and assume that all of the coefficients are smooth.

The method of characteristics

Define the **characteristic**, or **streamline**, to be the solution $x(s)$ to the system of ordinary differential equations

$$\frac{dx_j(s)}{ds} = a_j(x(s)).$$

This gives us a set of coordinates $x(s)$ for every set of initial values $x(0)$.

Now back to our problem.

$$u_t + a \cdot \bigtriangledown u + a_0 u = f$$

Let $w(s) = u(x(s), s)$. Then the Chain Rule tells us that

$$\frac{dw}{ds} = u_t + \bigtriangledown u \cdot \frac{dx}{ds} = u_t + a \cdot \bigtriangledown u,$$

so our problem becomes

$$w_s + a_0 \mathbf{w} = f$$

with $w(0) = v(x_0)$ for each x_0 on the inflow boundary.

This is just an IVP ordinary differential equation. Most amazingly, **the solution along the entire characteristic depends only on the single initial value $v(x_0)$ at the point where the characteristic starts**.

Examples: See p.171-3.

Jargon:

- Points at which the characteristics enter Ω are part of the **inflow boundary**. (Γ_- in the book)
- Points at which the characteristics leave Ω are part of the **outflow boundary**. (Γ_+)
- Points for which $n(x) \cdot a(x) = 0$ are on the **characteristic boundary** (Γ_0)

Symmetric hyperbolic systems

Reference: Section 11.4

For $x \in \mathcal{R}$ and $t \geq 0$, let $u, f : \mathcal{R}^2 \to \mathcal{R}^n$ satisfy

$$u_t + A(x,t)u_x + B(x,t)u = f(x,t)$$

with initial values $u(x, 0) = v(x)$.

Suppose A, B, and f are smooth functions.

If $A = P\Lambda P^T$ is symmetric with distinct eigenvalues λ_j $(j = 1, \dots, n)$ then we say that the system is **strictly hyperbolic**.

P is the matrix with the eigenvectors of A as its columns, so $P^T P = I$.

$$u_t + A(x,t)u_x + B(x,t)u = f(x,t)$$

Let's change variables: $w = P^T u$ and multiply our equation by P^T. Then

$$\mathbf{P^T u_t} + \Lambda \mathbf{P^T u_x} + P^T B u = P^T f.$$

Now $u = Pw$, so

$$\begin{aligned} \mathbf{P^T u_t} &= P^T(Pw_t + P_t w) = \mathbf{w_t + P^T P_t w} \\ \mathbf{P^T u_x} &= P^T(Pw_x + P_x w) = \mathbf{w_x + P^T P_x w} \end{aligned}$$

so our equation becomes

$$\mathbf{w_t + P^T P_t w} + \Lambda(\mathbf{w_x + P^T P_x w}) + P^T B P w = P^T f,$$

or

$$\mathbf{w_t} + \Lambda \mathbf{w_x} + (P^T B P + \mathbf{P^T P_t} + \Lambda \mathbf{P^T P_x}) w = P^T f \equiv \tilde{f}$$

which looks like the original equation except that the w_x coefficient is diagonal.

$$w_t + \Lambda w_x + \tilde{B} w = \tilde{f}$$

Case 1: $\tilde{B} = 0$. Then we have n uncoupled ODEs

$$(w_j)_t + \lambda_j (w_j)_x = \tilde{f}_j,$$

with $w_j(x,0)$ given.

We know how to solve these equations using the **method of characteristics**

Once we have the solution, we form $u = Pw$ and we are done.

Therefore, we have existence of the solution. Well-posedness also holds.

Example: p177.

Case 2: $\tilde{B} \neq 0$. Then we can't explicitly write the solution, but your book shows that the problem is well posed.

Conclusion

We have defined well-posed problems in three forms:

- The wave equation, analyzed by eigenfunctions of $-\Delta u$.
- First order scalar equations, analyzed by characteristics.
- Symmetric hyperbolic systems, decoupled by matrix eigenvectors and solved by characteristics.

Next: Numerical methods.