AMSC/CMSC 661 Scientific Computing II
Spring 2005
Solution of Elliptic PDEs
Part 3

Dianne P. O'Leary

These notes are based on the 2003 textbook of Stig Larsson and Vidar Thomée.

The Elliptic Eigenvalue Problem

The importance of this section lies in three disjoint uses:

- Sometimes, eigenvalue problems arise in applications. For example, we might be interested in the natural frequencies of vibration of a bridge or membrane.
- Eigenfunction expansions can be used to solve elliptic PDEs (spectral methods).
- The theory we develop is the basis for the Fourier / Wavelet discussion we take up at the end of the course.

The plan

- What is an eigenvalue problem?
- Digression: properties of an orthonormal basis for a space.
- How do we solve an eigenproblem numerically?

What is an eigenvalue problem?

Review: Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Notice that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \end{bmatrix}.$$

In other words, we have found 4 vectors, called **eigenvectors** of A, that have the special property that multiplication by A just scales the vector.

We call the scale factor the **eigenvalue** of A, and we can abbreviate the relation

$$A\phi_j = \lambda_j \phi_j$$

where, in our example, $\lambda_1 = -4, \lambda_2 = 1, \lambda_3 = 2$, and $\lambda_4 = 3$ are the eigenvalues and the eigenvectors ϕ_j are the unit vectors.

Some properties of eigenvectors

- When the eigenvalues are distinct, the eigenvectors are unique, except that they can be scaled by any nonzero number. We will assume that $\|\phi_j\| = 1$.
- The eigenvectors are also **linearly independent**, so they form a basis for $\mathcal{R}^n$.
- In fact, if A is symmetric, then eigenvectors corresponding to distinct eigenvalues are orthogonal.
- **Jargon:** If all of the eigenvalues are positive, we say that A is **positive definite**.
- The smallest eigenvalue λ_1 is the value of the function

  $$\min_{x \neq 0} \frac{x^T A x}{x^T x}$$

  and this value is achieved for $x = \phi_1$
- The other eigenvalues can also be characterized as solutions to minimization problems (or maximization problems).

The elliptic eigenvalue problem

Now, as an example, let

$$\mathcal{A}u = -u'',$$

and require $u(0) = u(1) = 0$.

Notice that for $j = 1, 2, \ldots,$

$$\mathcal{A} \sin(j\pi x) = (j\pi)^2 \sin(j\pi x).$$

In other words, we **have** found functions $\phi_j = \sin(j\pi x)$, called **eigenfunctions** of $\mathcal{A}$, that satisfy the boundary conditions and have the special property that applying $\mathcal{A}$ just scales the function. We call the scale factor the **eigenvalue** of $\mathcal{A}$, and we can abbreviate the relation as

$$\mathcal{A}\phi_j = \lambda_j \phi_j$$

where $\lambda_j = (j\pi)^2$.

All of the properties that we listed for eigenvectors also hold for eigenfunctions.

Formulating the elliptic eigenvalue problem

Strong formulation: Given $\mathcal{A}$ and homogeneous Dirichlet boundary conditions, find numbers λ and functions $\phi \in H_0^1$ satisfying

$$\mathcal{A}\phi = \lambda\phi.$$

We'll assume in this chapter that the operator is just $\mathcal{A}\phi = -\bigtriangledown \cdot (a \bigtriangledown \phi) + c\phi$.

(In fact, your book assumes $\mathcal{A}\phi = -\bigtriangledown^2 \phi$.)

Weak formulation: Given $\mathcal{A}$, find numbers λ and functions $\phi \in H_0^1$ satisfying

$$a(\phi, v) = \lambda(\phi, v)$$

for all $v \in H_0^1$, where $a(\phi, v)$ and (ϕ, v) are defined as before.

Some properties

Theorem 6.1: (p. 79)
6.1a: The eigenvalues of $\mathcal{A}$ are positive.
6.1b: If $\mathcal{A}\phi = \lambda\phi$ and $\mathcal{A}\psi = \nu\psi$ and $\lambda \neq \nu$, then $(\phi, \psi) = 0$.

Proof:
6.1a: Suppose $\mathcal{A}\phi = \lambda\phi$. Then

$$0 < a(\phi, \phi) = \lambda(\phi, \phi)$$

so $\lambda > 0$.

6.1b:

$$\begin{aligned} a(\phi, \psi) &= \lambda(\phi, \psi), \\ a(\psi, \phi) &= \nu(\psi, \phi), \end{aligned}$$

so, subtracting,

$$0 = (\lambda - \nu)(\phi, \psi)$$

which forces $(\phi, \psi) = 0$. []

Digression: Orthonormal bases

Begin Digression: So the eigenfunctions are orthogonal. Let's normalize them ($\|\phi_j\| = 1$) and see what orthogonality + normalization = **orthonormality** tells us.

Definition: Suppose the $\{\phi_j\}$ are orthonormal and are in a space H. Then they form an **orthonormal basis** for H if, given any $v \in H$ and any $\epsilon > 0$ there exist coefficients a_j and an integer N so that

$$\|v - \sum_{j=1}^{N} a_j \phi_j\| < \epsilon.$$

We need a recipe for finding the coefficients a_j.

Lemma 6.1a: (p. 82) The best approximation to $v \in H$ by the first N functions in the orthonormal basis $\{\phi_j\}$ is

$$v_N = \sum_{j=1}^{N} (v, \phi_j)\phi_j.$$

Proof: Take any approximation, and compute the residual it makes:

$$\begin{aligned}
\|v - \sum_{j=1}^{N} a_j \phi_j\|^2 &= (v - \sum_{j=1}^{N} a_j \phi_j, v - \sum_{k=1}^{N} a_k \phi_k) \\
&= (v,v) - 2\sum_{j=1}^{N} a_j (v, \phi_j) + \sum_{k=1}^{N}\sum_{j=1}^{N} a_k a_j (\phi_\mathbf{k}, \phi_\mathbf{j}) \\
&= (v,v) - 2\sum_{j=1}^{N} a_j (v, \phi_j) + \sum_{j=1}^{N} a_j^2 \\
&= (v,v) + \sum_{j=1}^{N} (a_j - (v,\phi_j))^2 - \sum_{j=1}^{N} (v,\phi_j)^2,
\end{aligned}$$

and we make this as small as possible by making the middle term zero, setting $a_j = (v, \phi_j)$. []

Lemma 6.1b: Bessel's inequality (p. 82) For all $v \in H$, if $\{\phi_j\}$ is **a set of orthonormal functions** in H, then

$$\sum_{j=1}^{\infty} (v, \phi_j)^2 \le \|v\|^2.$$

Proof: For our choice of coefficients $a_1, \ldots, a_N$,

$$0 \le \|v - \sum_{j=1}^{N} a_j \phi_j\|^2 = (v,v) + \sum_{j=1}^{N} (a_j - (v,\phi_j))^2 - \sum_{j=1}^{N} (v,\phi_j)^2 = (v,v) - \sum_{j=1}^{N} (v,\phi_j)^2,$$

so

$$\sum_{j=1}^{N} (v,\phi_j)^2 \le (v,v).$$

Take the limit as $N \to \infty$. []

Lemma 6.1c: Parseval's Relation (p. 82) For all $v \in H$, if $\{\phi_j\}$ is **an orthonormal basis** for H, then

$$\sum_{j=1}^{\infty} (v,\phi_j)^2 = \|v\|^2.$$

Proof: For an orthonormal basis, the v_N gets arbitrarily close to v, so the norm must converge. []

End of digression.

A few more facts about eigenvalues and eigenfunctions

The following facts are proven in your book when $\mathcal{A} = -\nabla^2$, but we will take them on faith for all elliptic operators:

- If $\mathcal{A}$ has an infinite number of eigenvalues, then $\lambda_n \to \infty$. (Thm. 6.3, p. 81)
- If $\mathcal{A}$ has an infinite number of eigenvalues, then the eigenfunctions form an orthonormal basis for L_2, and

  $$a(v,v) = \sum_{j=1}^{\infty} \lambda_j (v,\phi_j)^2 < \infty$$

  if and only if $v \in H_0^1$. (Thm. 6.4, p. 83)
- Min-Max Characterization of Eigenvalues:

  $$\lambda_n = \min_{V_n} \max_{v \in V_n} \frac{a(v,v)}{(v,v)}$$

  where V_n varies over all subspaces of H_0^1 of dimension n. (Thm. 6.5, p. 84)
- Monotonicity: If $\Omega \subset \tilde{\Omega}$, then $\lambda_n(\Omega) \ge \lambda_n(\tilde{\Omega})$. (p. 84)

Numerical solution of the elliptic eigenproblem

Idea:

- Replace $\mathcal{A}$ by $\mathcal{A}_h$, where $\mathcal{A}_h$ is the **finite difference** matrix or **finite element** stiffness matrix.
- Use the eigenvalues $\lambda_{n,h}$ of $\mathcal{A}_h$ as approximations to the smallest eigenvalues of $\mathcal{A}$.
- For finite differences, the eigenvectors of $\mathcal{A}_h$ contain approximate values of the eigenfunctions at the mesh points.
- For finite elements, the eigenvectors of $\mathcal{A}_h$ contain coefficients in an expansion of the eigenfunction in the finite element basis.

We'll just consider the finite element approximation.

Accuracy of the computed eigenvalues

Theorem 6.7: (p. 90) Notation:

- $\lambda_{n,h}$ = eigenvalue n of A_h.
- λ_n = eigenvalue n of $\mathcal{A}$.

There exist constants C and h_0, depending on n, such that when $h < h_0$,

$$\lambda_n \le \lambda_{n,h} \le \lambda_n + Ch^2.$$

Proof of 1st inequality:

$$\begin{aligned} \lambda_n &= \min_{V_n} \max_{v \in V_n} \frac{a(v,v)}{(v,v)} \\ &\le \min_{V_n \subset S_h} \max_{v \in V_n} \frac{a(v,v)}{(v,v)} \\ &= \lambda_{n,h}. \end{aligned}$$

We'll take the 2nd inequality on faith. []

Accuracy of the computed eigenfunctions

Theorem: (extension of Theorem 6.8, p. 92) Notation:

- $\phi_{n,h}$ = function obtained from the eigenvector of A_h (piecewise linear basis functions) corresponding to a **simple** eigenvalue $\lambda_{n,h}$.
- ϕ_n = eigenfunction of $\mathcal{A}$ corresponding to a **simple** eigenvalue λ_n.

There exist constants C and h_0, depending on n, such that when $h < h_0$,

$$\|\phi_{n,h} - \phi_n\| \le Ch^2.$$