15-213

"The course that gives CMU its Zip!"

Integers
Sep 3, 2002

Topics

- Numeric Encodings
  - Unsigned & Two's complement
- Programming Implications
  - C promotion rules
- Basic operations
  - Addition, negation, multiplication
- Programming Implications
  - Consequences of overflow
  - Using shifts to perform power-of-2 multiply/divide

C Puzzles

- Taken from old exams
- Assume machine with 32 bit word size, two's complement integers
- For each of the following C expressions, either:
  - Argue that is true for all argument values
  - Give example where not true

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

- `x < 0` ⇒ `((x*2) < 0)`
- `ux >= 0`
- `x & 7 == 7` ⇒ `(x<<30) < 0`
- `ux > -1`
- `x > y` ⇒ `-x < -y`
- `x * x >= 0`
- `x > 0 && y > 0` ⇒ `x + y > 0`
- `x >= 0` ⇒ `-x <= 0`
- `x <= 0` ⇒ `-x >= 0`

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- C `short` 2 bytes long

DecimalHexBinaryx			

Sign Bit

- For 2's complement, most significant bit indicates sign
  - 0 for nonnegative
  - 1 for negative

Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213	-15213

Numeric Ranges

Unsigned Values

- *UMin* = 0
  000…0
- *UMax* = $2^w - 1$
  111…1

Two's Complement Values

- *TMin* = -2^{w-1}
  100…0
- *TMax* = $2^{w-1} - 1$
  011…1

Other Values

- Minus 1
  111…1

Values for $W = 16$

DecimalHexBinaryUMax			

Values for Different Word Sizes

W8163264				

Observations

- $|TMin| = TMax + 1$
  - Asymmetric range
- $UMax = 2 * TMax + 1$

C Programming

- `#include <limits.h>`
  - K&R App. B11
- Declares constants, e.g.,
  - `ULONG_MAX`
  - `LONG_MAX`
  - `LONG_MIN`
- Values platform-specific

Unsigned & Signed Numeric Values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

Equivalence

- Same encodings for nonnegative values

Uniqueness

- Every bit pattern represents unique integer value
- Each representable integer has unique bit encoding

⇒ Can Invert Mappings

- U2B(x) = B2U^{-1}(x)
  - Bit pattern for unsigned integer
- T2B(x) = B2T^{-1}(x)
  - Bit pattern for two's comp integer

Casting Signed to Unsigned

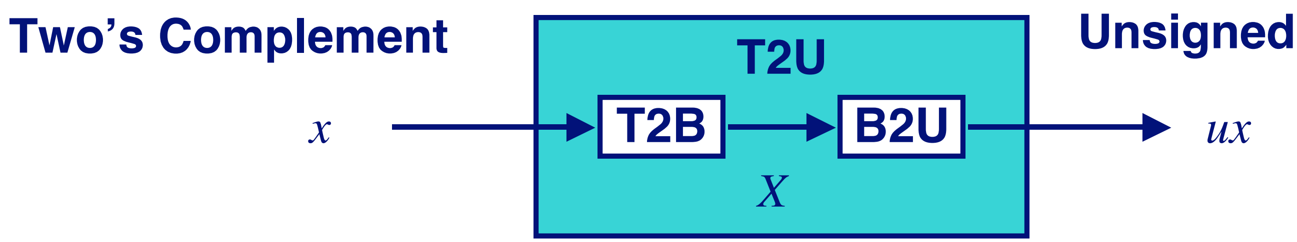
C Allows Conversions from Signed to Unsigned

```
short int          x =  15213;
unsigned short int ux = (unsigned short) x;
short int          y  = -15213;
unsigned short int uy = (unsigned short) y;
```

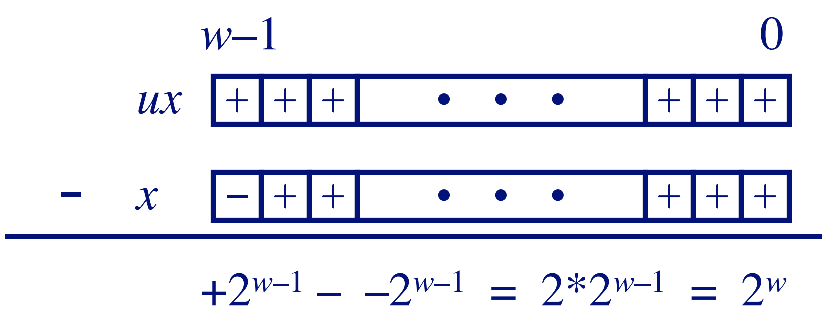
Resulting Value

- No change in bit representation
- Nonnegative values unchanged
  - *ux* = 15213
- Negative values change into (large) positive values
  - *uy* = 50323

Relation between Signed & Unsigned



Maintain Same Bit Pattern



$$ux = \begin{cases} x & x \ge 0 \\ x + 2^w & x < 0 \end{cases}$$

Relation Between Signed & Unsigned

Weight-1521350323		

- $uy \quad = \quad y + 2 * 32768 \quad = \quad y + 65536$

Signed vs. Unsigned in C

Constants

- By default are considered to be signed integers
- Unsigned if have "U" as suffix

```
0U, 4294967259U
```

Casting

- Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

- Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```

Casting Surprises

Expression Evaluation

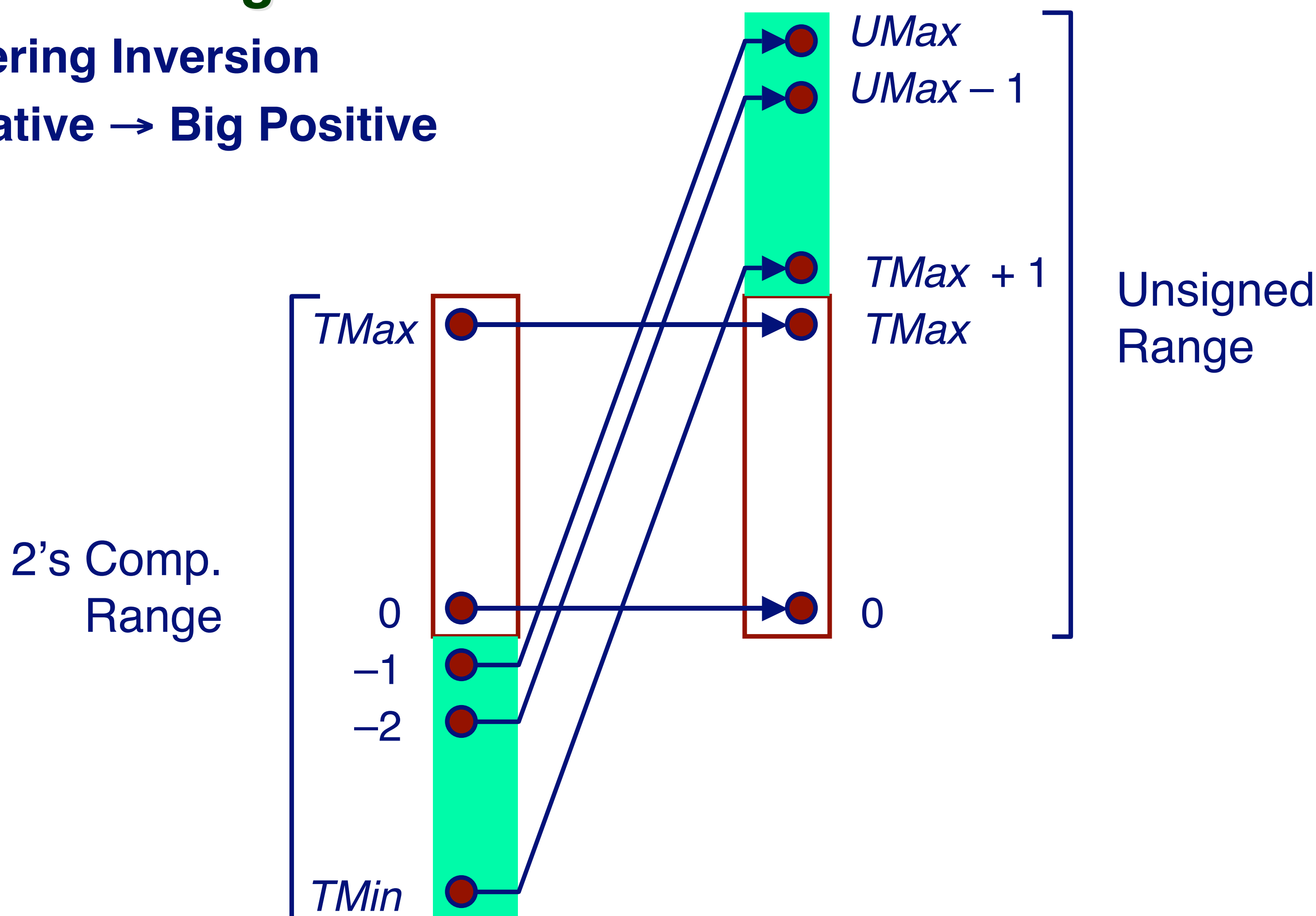
- If mix unsigned and signed in single expression, signed values implicitly cast to unsigned
- Including comparison operations <, >, ==, <=, >=
- Examples for $W = 32$

Constant$_1$	Constant$_2$	Relation	Evaluation
`0`	`0U`	==	unsigned
`-1`	`0`	<	signed
`-1`	`0U`	>	unsigned
`2147483647`	`-2147483648`	>	signed
`2147483647U`	`-2147483648`	<	unsigned
`-1`	`-2`	>	signed
`(unsigned) -1`	`-2`	>	unsigned
`2147483647`	`2147483648U`	<	unsigned
`2147483647`	`(int) 2147483648U`	>	signed

Explanation of Casting Surprises

2's Comp. → Unsigned

- **Ordering Inversion**
- **Negative → Big Positive**



Sign Extension

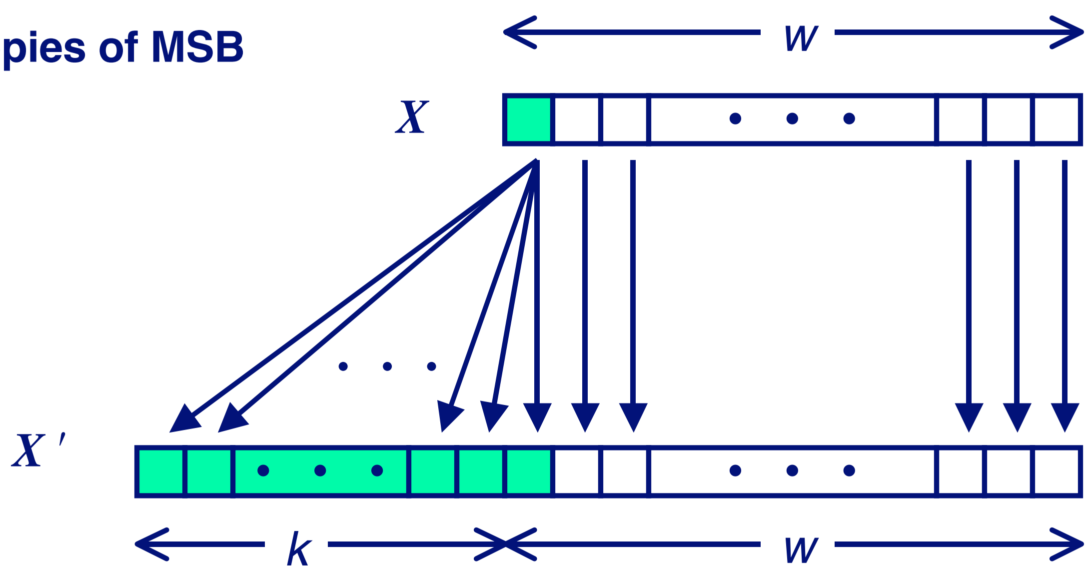
Task:

- Given *w*-bit signed integer *x*
- Convert it to *w*+*k*-bit integer with same value

Rule:

- Make *k* copies of sign bit:
- $X' = x_{w-1}, \ldots, x_{w-1}, x_{w-1}, x_{w-2}, \ldots, x_0$

k copies of MSB



Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

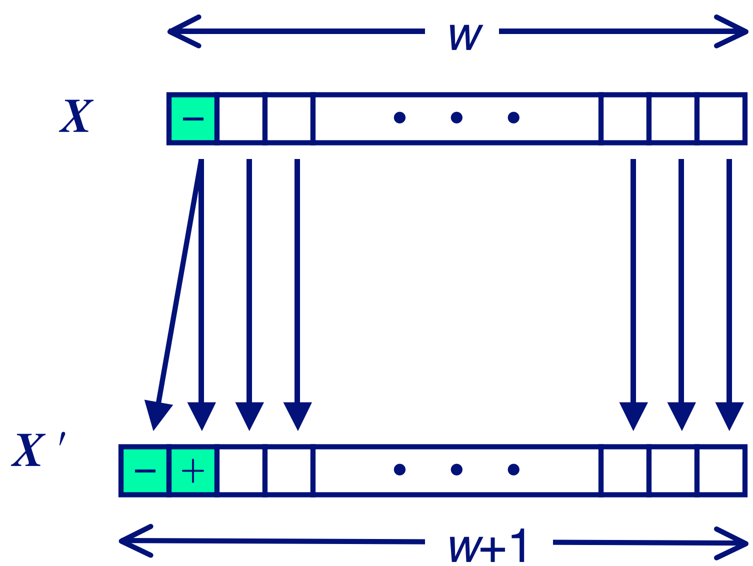
	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- **Converting from smaller to larger integer data type**
- **C automatically performs sign extension**

Justification For Sign Extension

Prove Correctness by Induction on *k*

- **Induction Step: extending by single bit maintains value**



- **Key observation:** $-2^{w-1} = -2^{w} + 2^{w-1}$
- **Look at weight of upper bits:**

  $X \quad -2^{w-1}\, x_{w-1}$

  $X' \quad -2^{w}\, x_{w-1} + 2^{w-1}\, x_{w-1} \quad = \quad -2^{w-1}\, x_{w-1}$

Why Should I Use Unsigned?

Don't Use Just Because Number Nonzero

- C compilers on some machines generate less efficient code

```
unsigned i;
for (i = 1; i < cnt; i++)
  a[i] += a[i-1];
```

- Easy to make mistakes

```
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

Do Use When Performing Modular Arithmetic

- Multiprecision arithmetic
- Other esoteric stuff

Do Use When Need Extra Bit's Worth of Range

- Working right up to limit of word size

Negating with Complement & Increment

Claim: Following Holds for 2's Complement

`~x + 1 == -x`

Complement

- Observation: `~x + x == 1111…11`$_2$ `== -1`

`x`	1	0	0	1	1	1	0	1
+ `~x`	0	1	1	0	0	0	1	0
`-1`	1	1	1	1	1	1	1	1

Increment

- `~x +` ~~`x`~~ `+ (`~~`-x`~~ `+ 1)    ==` ~~`-1`~~ `+ (-x +` ~~`1`~~`)`
- `~x + 1    ==    -x`

Warning: Be cautious treating `int`'s as integers

- OK here

Comp. & Incr. Examples

x = 15213

lHexBinaryx			

0

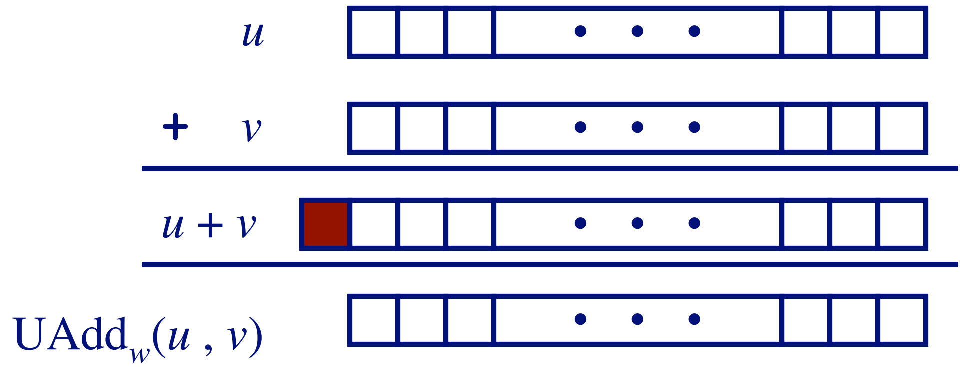
DecimalHexBinary0			~0

Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



Standard Addition Function

- **Ignores carry output**

Implements Modular Arithmetic

$s \;=\; \text{UAdd}_w(u\,,\,v) \;=\; u + v \bmod 2^w$

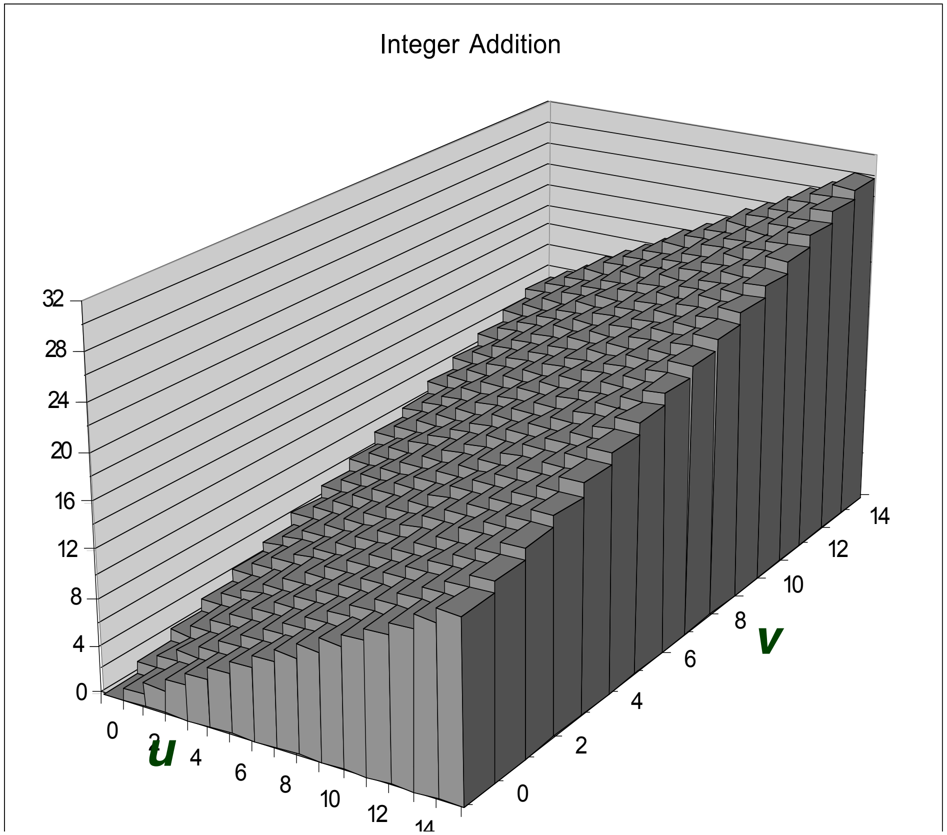
$$UAdd_w(u,v) = \begin{cases} u+v & u+v < 2^w \\ u+v-2^w & u+v \geq 2^w \end{cases}$$

Visualizing Integer Addition

Integer Addition

- 4-bit integers *u*, *v*
- Compute true sum $Add_4(u, v)$
- Values increase linearly with *u* and *v*
- Forms planar surface

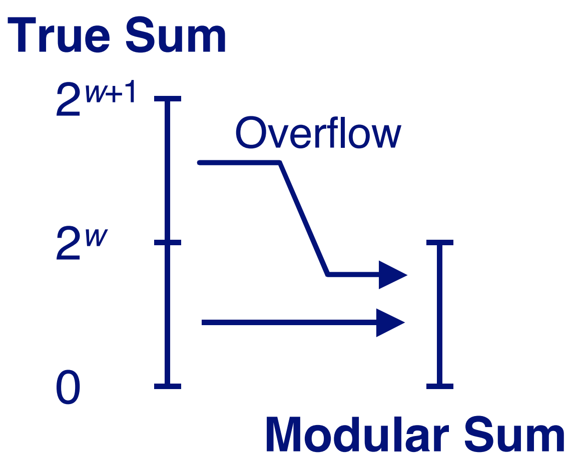
$Add_4(u, v)$

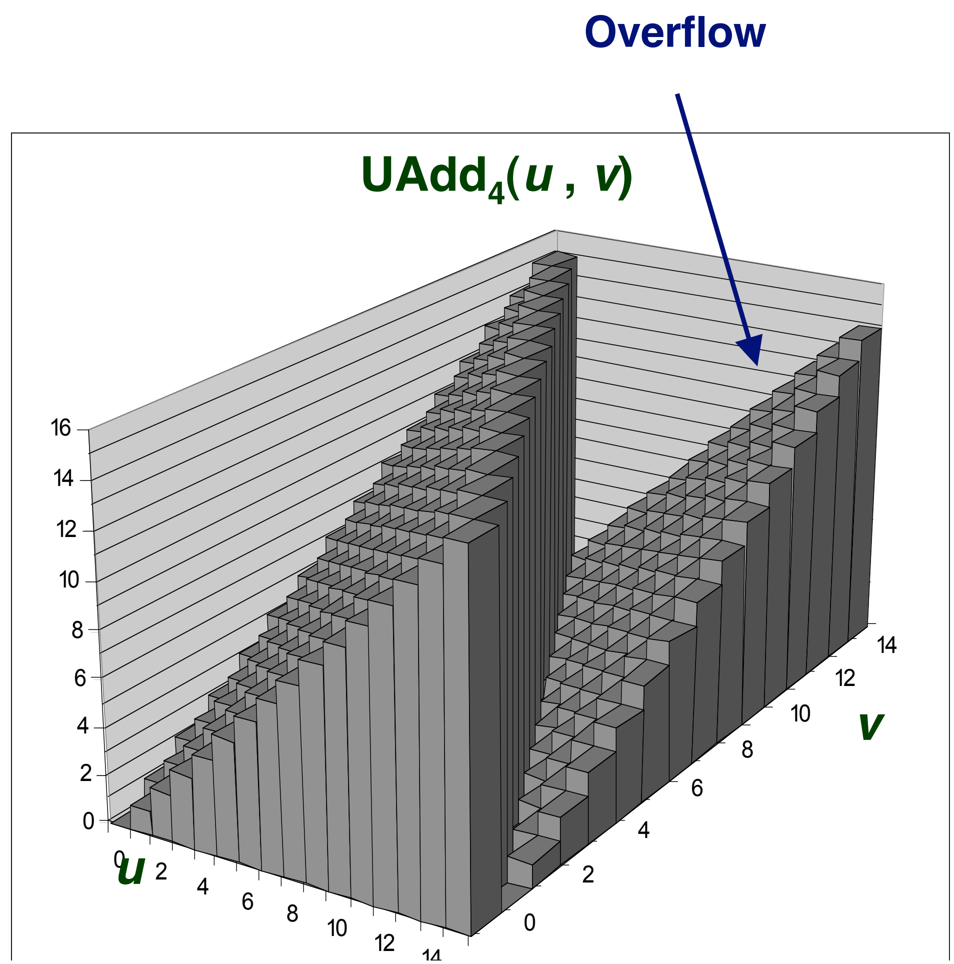


Visualizing Unsigned Addition

Wraps Around

- If true sum ≥ 2^w
- At most once





Mathematical Properties

Modular Addition Forms an *Abelian Group*

- **Closed under addition**

  $0 \leq \text{UAdd}_w(u, v) \leq 2^w - 1$

- **Commutative**

  $\text{UAdd}_w(u, v) = \text{UAdd}_w(v, u)$

- **Associative**

  $\text{UAdd}_w(t, \text{UAdd}_w(u, v)) = \text{UAdd}_w(\text{UAdd}_w(t, u), v)$

- **0 is additive identity**

  $\text{UAdd}_w(u, 0) = u$

- **Every element has additive inverse**
  - Let $\text{UComp}_w(u) = 2^w - u$

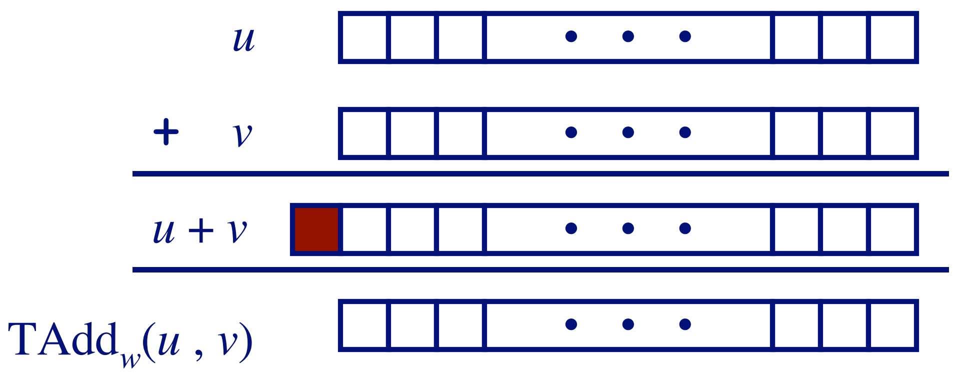
  $\text{UAdd}_w(u, \text{UComp}_w(u)) = 0$

Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



TAdd and UAdd have Identical Bit-Level Behavior

- **Signed vs. unsigned addition in C:**

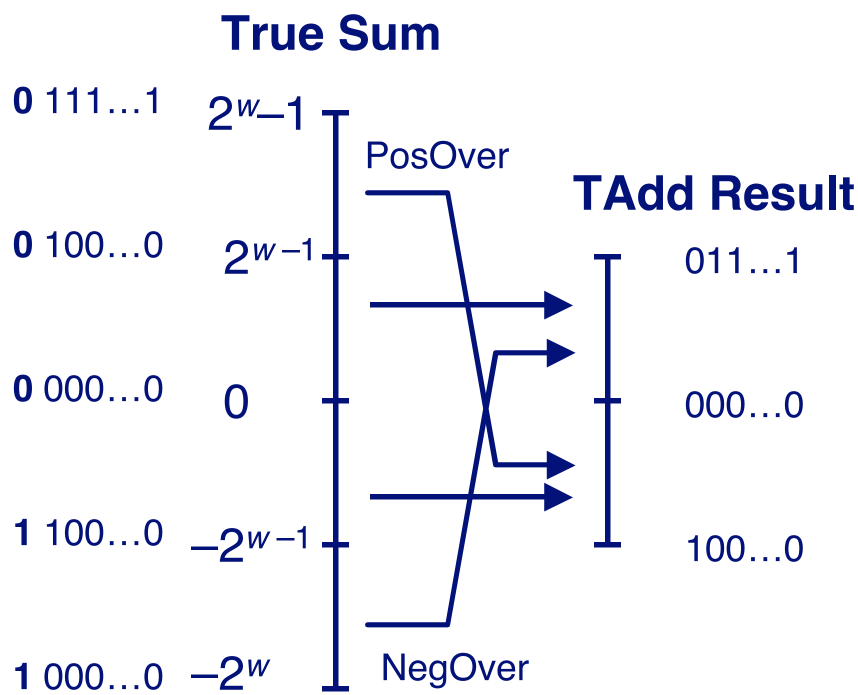
```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

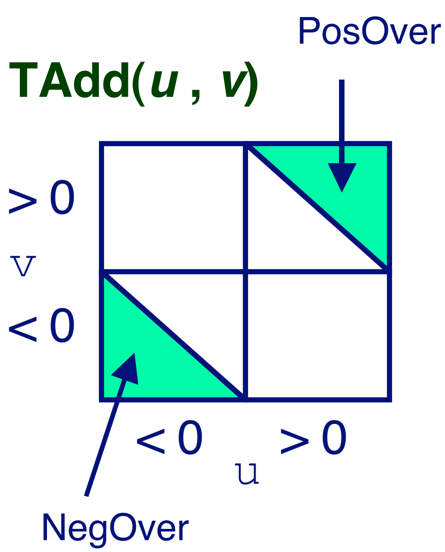
- **Will give** `s == t`

Characterizing TAdd

Functionality

- True sum requires w+1 bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer





$$TAdd_w(u,v) = \begin{cases} u+v+2^{w-1} & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^{w-1} & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$

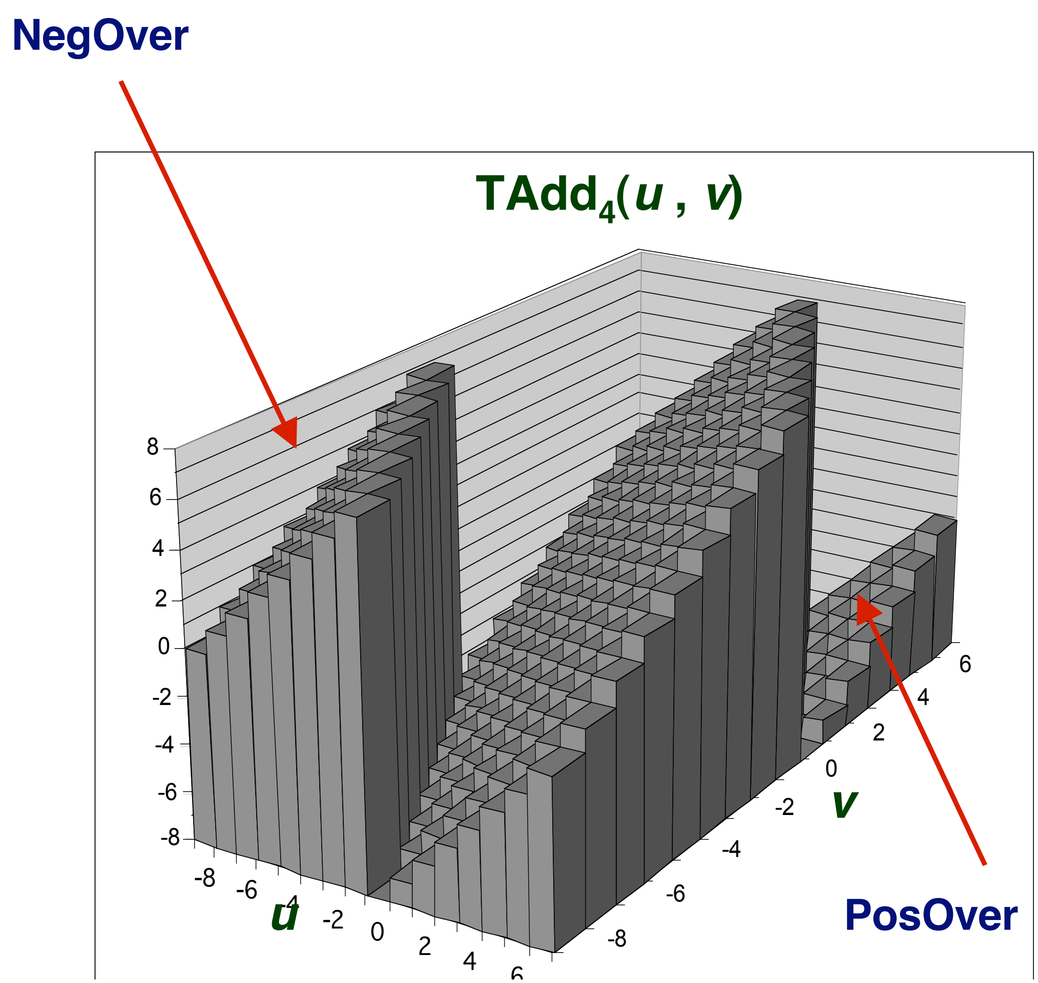
Visualizing 2's Comp. Addition

Values

- 4-bit two's comp.
- Range from -8 to +7

Wraps Around

- If sum $\geq 2^{w-1}$
  - Becomes negative
  - At most once
- If sum $< -2^{w-1}$
  - Becomes positive
  - At most once

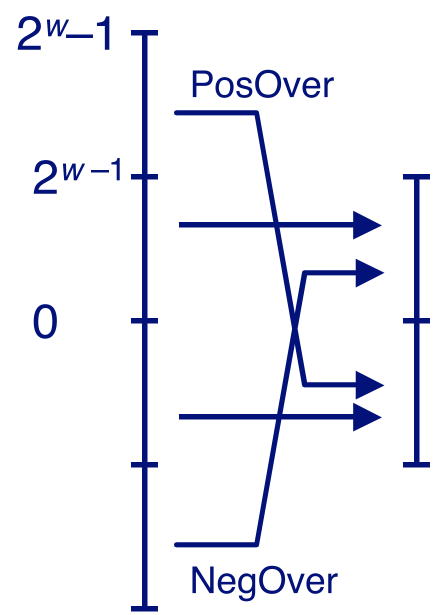


Detecting 2's Comp. Overflow

Task

- **Given** s = $TAdd_w(u, v)$
- **Determine if** s = $Add_w(u, v)$
- **Example**

```
int s, u, v;
s = u + v;
```



Claim

- **Overflow iff either:**

  ***u*, *v* < 0, *s* ≥ 0 (NegOver)**

  ***u*, *v* ≥ 0, *s* < 0 (PosOver)**

```
ovf = (u<0 == v<0) && (u<0 != s<0);
```

Mathematical Properties of TAdd

Isomorphic Algebra to UAdd

- $\text{TAdd}_w(u, v) = \text{U2T}(\text{UAdd}_w(\text{T2U}(u), \text{T2U}(v)))$
  - **Since both have identical bit patterns**

Two's Complement Under TAdd Forms a Group

- **Closed, Commutative, Associative, 0 is additive identity**
- **Every element has additive inverse**

  Let $\text{TComp}_w(u) = \text{U2T}(\text{UComp}_w(\text{T2U}(u)))$

  $\text{TAdd}_w(u, \text{TComp}_w(u)) = 0$

$$TComp_w(u) = \begin{cases} -u & u \neq TMin_w \\ TMin_w & u = TMin_w \end{cases}$$

Multiplication

Computing Exact Product of *w*-bit numbers *x*, *y*

- **Either signed or unsigned**

Ranges

- **Unsigned:** $0 \le x * y \le (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
  - **Up to 2*w* bits**
- **Two's complement min:** $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
  - **Up to 2*w*–1 bits**
- **Two's complement max:** $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
  - **Up to 2*w* bits, but only for $(TMin_w)^2$**

Maintaining Exact Results

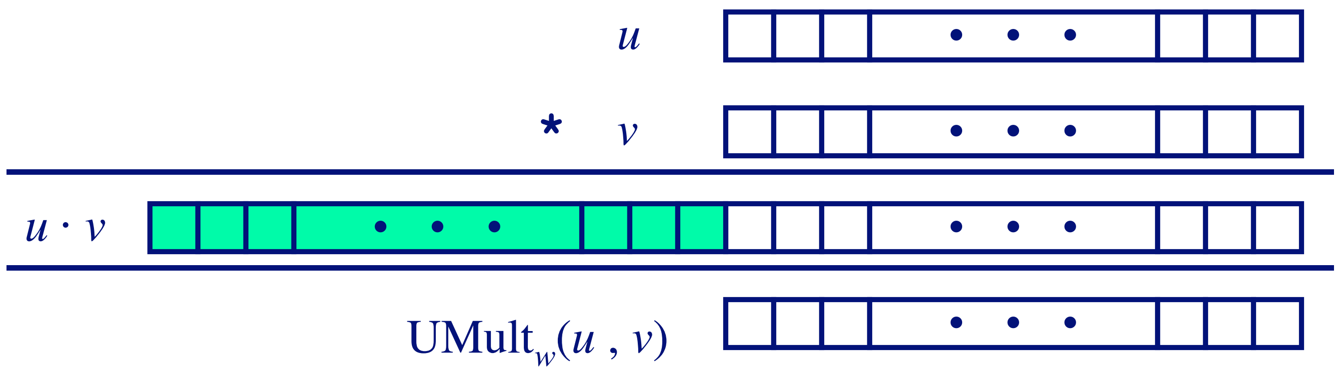
- **Would need to keep expanding word size with each product computed**
- **Done in software by "arbitrary precision" arithmetic packages**

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



Standard Multiplication Function

- **Ignores high order *w* bits**

Implements Modular Arithmetic

$\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$

Unsigned vs. Signed Multiplication

Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

- Truncates product to *w*-bit number $up = \text{UMult}_w(ux, uy)$
- Modular arithmetic: $up = ux \cdot uy \mod 2^w$

Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

- Compute exact product of two *w*-bit numbers *x*, *y*
- Truncate result to *w*-bit number $p = \text{TMult}_w(x, y)$

Unsigned vs. Signed Multiplication

Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

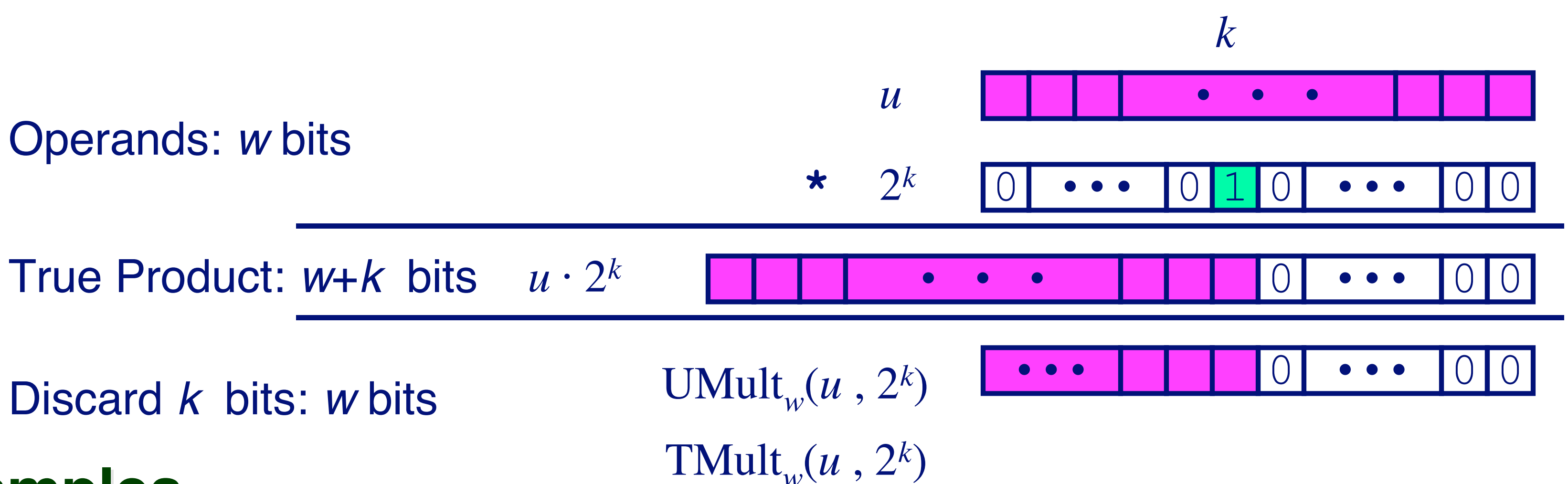
Relation

- Signed multiplication gives same bit-level result as unsigned
- `up == (unsigned) p`

Power-of-2 Multiply with Shift

Operation

- `u << k` gives `u *` 2^k
- Both signed and unsigned



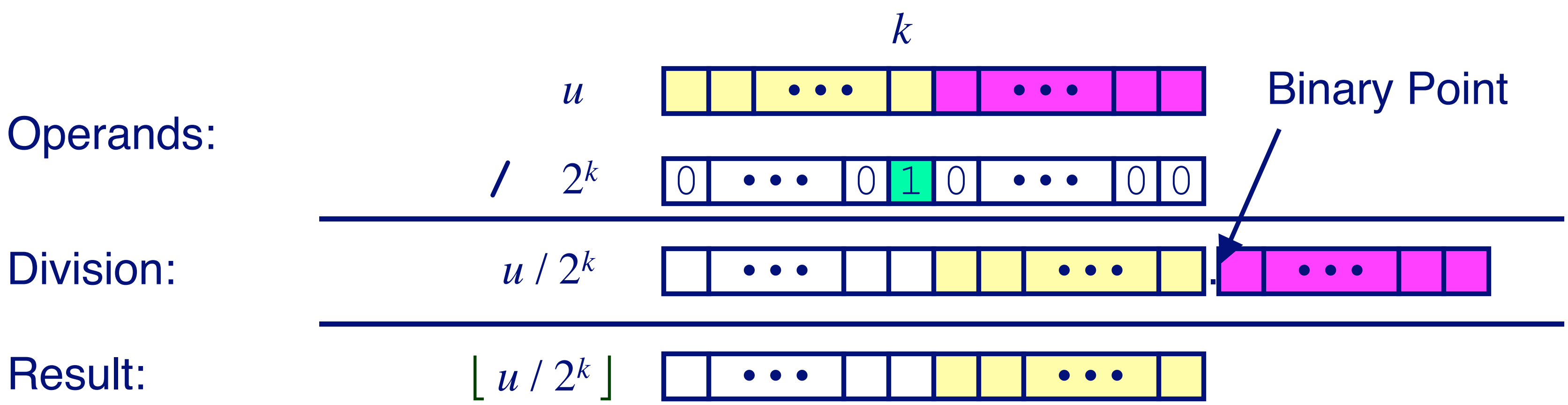
Examples

- `u << 3          ==  u * 8`
- `u << 5 - u << 3     ==     u * 24`
- Most machines shift and add much faster than multiply
  - Compiler generates this code automatically

Unsigned Power-of-2 Divide with Shift

Quotient of Unsigned by Power of 2

- `u >> k` gives $\lfloor$ `u / ` $2^k \rfloor$
- Uses logical shift

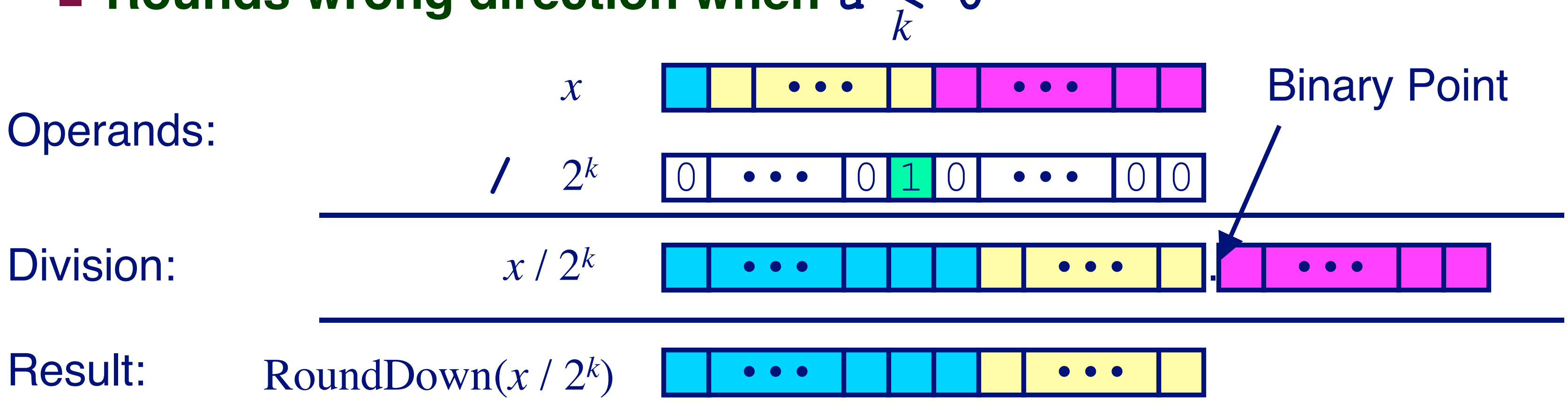


DivisionComputedHexBinaryx				x >

Signed Power-of-2 Divide with Shift

Quotient of Signed by Power of 2

- `x >> k` gives $\lfloor$ `x / ` $2^k \rfloor$
- Uses arithmetic shift
- Rounds wrong direction when `u < 0`



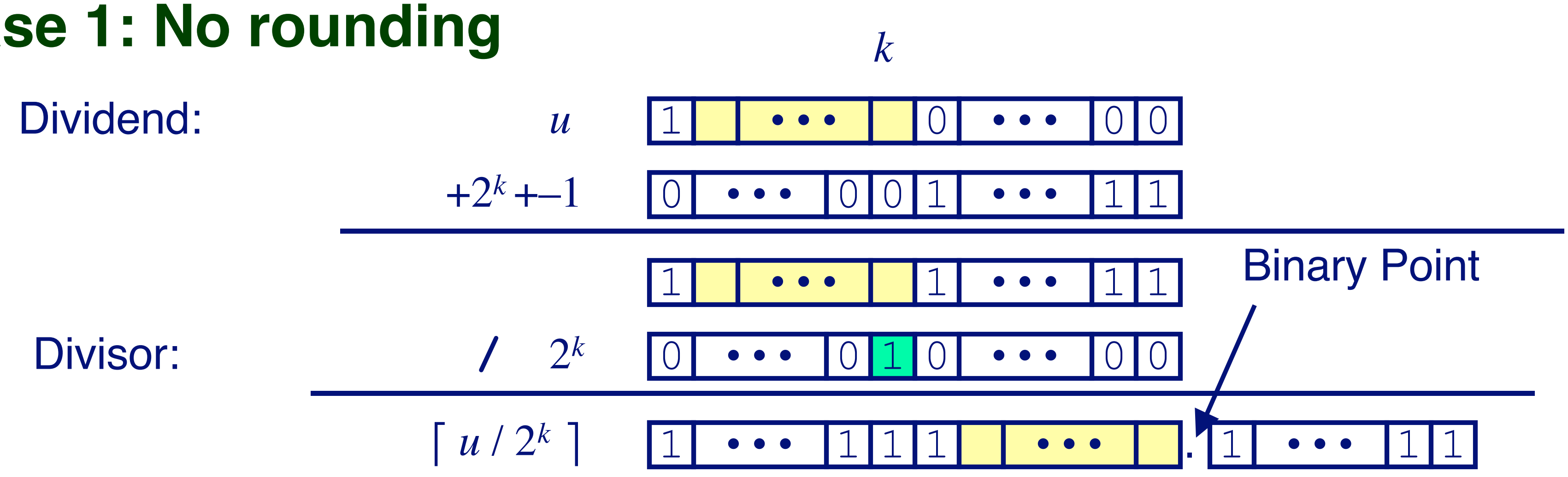
DivisionComputedHexBinary		y		y

Correct Power-of-2 Divide

Quotient of Negative Number by Power of 2

- **Want ⌈ `x / 2`^{k} ⌉ (Round Toward 0)**
- **Compute as ⌊ `(x+2`^{k}`-1) / 2`^{k} ⌋**
  - **In C: `(x + (1<<k)-1) >> k`**
  - **Biases dividend toward 0**

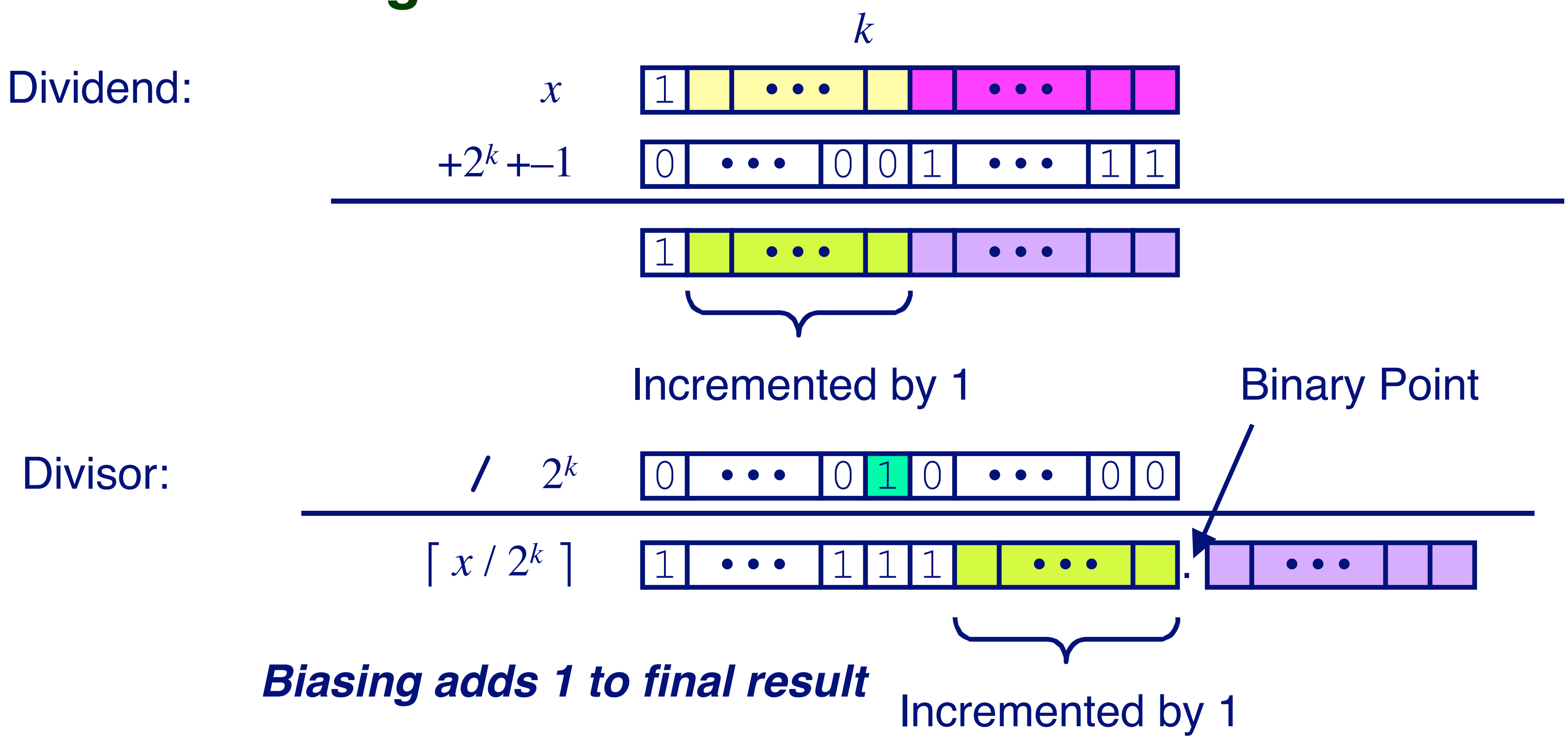
Case 1: No rounding



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

Properties of Unsigned Arithmetic

Unsigned Multiplication with Addition Forms Commutative Ring

- **Addition is commutative group**
- **Closed under multiplication**

  $0 \leq UMult_w(u, v) \leq 2^w - 1$
- **Multiplication Commutative**

  $UMult_w(u, v) = UMult_w(v, u)$
- **Multiplication is Associative**

  $UMult_w(t, UMult_w(u, v)) = UMult_w(UMult_w(t, u), v)$
- **1 is multiplicative identity**

  $UMult_w(u, 1) = u$
- **Multiplication distributes over addtion**

  $UMult_w(t, UAdd_w(u, v)) = UAdd_w(UMult_w(t, u), UMult_w(t, v))$

Properties of Two's Comp. Arithmetic

Isomorphic Algebras

- Unsigned multiplication and addition
  - Truncating to w bits
- Two's complement multiplication and addition
  - Truncating to w bits

Both Form Rings

- Isomorphic to ring of integers mod 2^w

Comparison to Integer Arithmetic

- Both are rings
- Integers obey ordering properties, e.g.,

  $u > 0 \quad \Rightarrow \quad u + v > v$

  $u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$

- These properties are not obeyed by two's comp. arithmetic

  TMax `+ 1` `==` *TMin*

  `15213 * 30426` `==` `-10030` (16-bit words)

C Puzzle Answers

- Assume machine with 32 bit word size, two's comp. integers
- *TMin* makes a good counterexample in many cases

`x < 0`	⇒	`((x*2) < 0)`	**False:** *TMin*
`ux >= 0`			**True:** 0 = *UMin*
`x & 7 == 7`	⇒	`(x<<30) < 0`	**True:** $x_1 = 1$
`ux > -1`			**False:** 0
`x > y`	⇒	`-x < -y`	**False:** `-1`, *TMin*
`x * x >= 0`			**False:** `30426`
`x > 0 && y > 0`	⇒	`x + y > 0`	**False:** *TMax*, *TMax*
`x >= 0`	⇒	`-x <= 0`	**True:** $-TMax < 0$
`x <= 0`	⇒	`-x >= 0`	**False:** *TMin*