Constructive Induction on Fib Runtime

Note Title

$T(0) = 0$

$T(1) = 1$

$T(n) = T(n-1) + T(n-2) + 1$

Find the smallest $x > 1$ s.t. $T(n) \leq x^n$

IndHyp $\forall 0 \leq i < k \quad T(i) \leq x^i$

$k \geq 2$

IndStep

Show $T(k) \overset{?}{\leq} x^k$

$T(k)$ ⇓ substitute

$T(k-1) + T(k-2) + 1 \overset{?}{\leq} x^k$

↳ by IH $\leq x^{k-1} + x^{k-2} + 1 \overset{?}{\leq} x^k$

New Goal

$$x^{k-1} + x^{k-2} + 1 \overset{?}{\leq} x^k$$

Divide both sides by x^{k-2}

$$x + 1 + \frac{1}{x^{k-2}} \overset{?}{\leq} x^2$$

Since our x will be ≥ 1 and $k-2$ is positive

$$\frac{1}{x^{k-2}} \leq 1$$

So....

$$x+1+\underbrace{\frac{1}{x^{k-2}}}_{\leq 1} < \underbrace{x+1+1 \overset{?}{\leq} x^2}_{\text{New Goal}}$$

$$x + 2 \overset{?}{\leq} x^2$$

Solve using algebra...