Regular Graph Properties
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1 Introduction

Consider the following question:

Is the set of Hamiltonian graphs regular?

To pose this question properly we need to specify (a) which strings represent graphs, (b) what do to if a string that does not represent a graph is input.

Def 1.1 All strings are over the alphabet $\{0, 1, \$\}$. Let x be a string of the form $\$x_1\$x_2\$\cdots\$x_n\$$ where the following happen:

1. $(\forall i)[x_i \in \{0, 1\}^n]$. We will let

$$x_i = x_{i1} \cdots x_{in}.$$

2. View the string as the following matrix:

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

3. For all $1 \le i < j \le n$, $x_{ij} = x_{ji}$.

4. For all $1 \le i \le n$, $x_{ii} = 0$.

Any string of the form above is interpreted as the adjacency matrix of a graph.

We identify a graph G with the string that is its adjacency matrix, as above. Hence we will say things like *Run DFA M on G.*

Example 1.2 The graph K_4 is the string

$$\$0111\$1011\$1101\$1110\$$$

Def 1.3 Let $\mathcal{G}$ be a set of graphs.

1. Let $\mathcal{G}$ is a *graph property* if $\mathcal{G}$ satisfies the following property: for all pairs of graphs, (G_1, G_2), if G_1 and G_2 are isomorphic then either $G_1, G_2 \in \mathcal{G}$ or $G_1, G_2 \notin \mathcal{G}$.

2. A graph property $\mathcal{G}$ is *regular* if there exists a DFA M such that the following hold:

   (a) If $G \in \mathcal{G}$ then $M(G)$ accepts.

   (b) If $G \notin \mathcal{G}$ then $M(G)$ rejects.

3. If w is a string that does not represent an adjacency matrix then we have *no condition* on what $M(w)$ is.

2 Graph Properties that are Regular

Def 2.1

1. If $w \in \{0,1\}^*$ then $\#_1(w)$ is the number of 1's in w.

2. Let $A \subseteq \mathsf{N}$. A is *regular* if the following set is regular:

$$\{w \in \{0,1\}^* : \#_1(w) \in A\}$$

   is regular.

Theorem 2.2 *Let $A \subseteq \mathsf{N}$ be regular.*

1. *The following graph property is regular:*

$$\mathcal{G} = \{G = (V, E) : (\forall v \in V)[|\deg(v)| \in A]\}.$$

2. *The following graph property is regular:*

$$\mathcal{G} = \{G = (V, E) : |E| \in A\}.$$

Proof:
1) Let $W = \{w \in \{0,1\}^* : \#_1(w) \in A\}$. W is regular by the definition of $A \subseteq \mathsf{N}$ being regular. Let α be the regular expression such that $L(W) = \alpha$.

A graph G is in $\mathcal{G}$ iff every row of its adjacency matrix is in L. Consider the regular expression

$$\beta = \$(\alpha\$)^*$$

It is easy to see that
$G \in \mathcal{G}$ implies $G \in L(\beta)$.
$G \notin \mathcal{G}$ implies $G \notin L(\beta)$.

2) We leave this to the reader.
▮

Corollary 2.3 *Let $d, m \in 2$. The following graph properties are regular.*

1. *The set of graphs where* $(\forall v)[\deg(v) \geq d]$.
2. *The set of graphs where* $(\forall v)[\deg(v) = d]$.
3. *The set of graphs where* $(\forall v)[\deg(v) \leq d]$.
4. *Let $A \subseteq \{0, 1, \ldots, d-1\}$. The set of graphs where*

$$\{\deg(v) \pmod{m} : v \in V\} \subseteq A\}.$$

5. *The set of Eulerian graphs. (This is the $m = 2$, $A = \{0\}$ case.)*
6. *The set of graphs $G = (V, E)$ such that $|E| \equiv d \pmod{m}$.*

Open Problem 2.4 Aside from Eulerian graphs and (arguably) the set of graphs of constant degree, are there any other *interesting* graph properties that are regular.

3 Graph Properties that are not Regular

Theorem 3.1 *Let $\mathcal{G}$ be a graph property. Assume there exists $n, d \in \mathsf{N}$ and graphs G_1, G_2 such that the following hold:*

1. *G_1 and G_2 have n vertices $\{1, \ldots, n\}$.*
2. *$\deg_{G_2}(1) - \deg_{G_1}(1) \geq d$.*
3. *$G_1 - \{1\} \equiv G_2 - \{1\}$.*
4. *$G_1 \in \mathcal{G}$ and $\mathcal{G}_2 \notin \mathcal{G}$*

Then $\mathcal{G}$ is not regular.

Proof:

Assume $\mathcal{G}$ is regular via DFA M. Assume that M has n states. Let m be a number to be picked later; however, it is large. Run M on the strings

$\$0^m1^0\$$
$\$1^10^{m-1}\$$
$\$1^20^{m-2}\$$
$\vdots$
$\$1^m0^0\$$

Let m be so large that there exists $0 \leq m_1 < m_2 \leq m$ such that

1. $M(1^{m_1}0^{m-m_1}) = M(1^{m_2}0^{m-m_2}) = p$ (p is a state in M).
2. $m_2 - m_1 \geq d$

$\blacksquare$