Talks and Projects For the Ramsey Gang

Every page has a new project.
Some of the pages will have slides from a talk about the topic.
Some of the pages will have a recording of a talk about the topic.
Some will have a pointer to notes about the topic.
Some will have project ideas.
Some will be written before I talk about it to you.

1 Rectangle Free Colorings of Grids

Slides:

https://www.cs.umd.edu/~gasarch/COURSES/250/S24/slides/COMB/gridtalk.pdf

Recording: There is no recording.

Project Ideas:

I proved that a grid is 2-colorable iff it does NOT have 5×5 or 3×7 or 7×3 inside it.

1. Obtain this result with a SAT Solver or a program rather than by clever math.
2. Obtain the obstruction set for 3-coloring. (This is known.)
3. Obtain the obstruction set for 4-coloring. (This is known.)
4. Obtain the obstruction set for 5-coloring. (This is NOT known.)
5. Instead of rectangles look at squares. (It is known that for every c-coloring of some large enough grid there is a mono square. For $c = 2$ its known. Beyond that, unknown. Obstructs sets are unknown.)
6. Instead of rectangles look at right triangles. You may also demand they be similar to a particular right triangle with natural number sides (e.g., 3-4-5). I do not know whats known here. See later project about Euclidean Ramsey Theory.

2 Lower Bounds on Equational Ramsey Theory

Pitch: The following is known:

1) There is a 2-coloring COL of $\{1, \ldots, 7824\}$ such that THERE IS NO x, y, z such that (a) $x, y, z \in \{1, \ldots, 7824\}$ (b) $\mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$, (c) $x^2 + y^2 = z^2$.

2) For all 2-colorings COL of $\{1, \ldots, 7825\}$ THERE EXISTS x, y, z such that (a) $x, y, z \in \{1, \ldots, 7825\}$ (b) $\mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$, (c) $x^2 + y^2 = z^2$.

This result was achieved with a supercomputer. See `https://en.wikipedia.org/wiki/Boolean_Pythagorean_triples_problem` for more details on how much computer time and space was used.

What can we done without such large and expensive computers? We won't be able to prove Part 2, but perhaps we can prove a weaker version of part 1.
Find a large n such that there is a 2-coloring COL *of* $\{1, \ldots, n\}$ *such that THERE IS NO* x, y, z *such that (a)* $x, y, z \in \{1, \ldots, n\}$ *(b)* $\mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$*, (c)* $x^2 + y^2 = z^2$.

How to do this.

METHOD ONE: Greedy

Intuition: when trying to color z, use the least color that won't cause a problem.
$\mathrm{COL}(1) = 1$.

When trying to color z, we say that a color c is NOT ALLOWED if there is an x, y with $x^2 + y^2 = z^2$ and $\mathrm{COL}(x) = \mathrm{COL}(y) = c$. We will say color c is ALLOWED otherwise.
For $z = 2$ to ∞ (Assume $1, \ldots, z-1$ have been colored.)

1. If 1 is allowed then set $\mathrm{COL}(z) = 1$.
2. If 1 is not allowed but 2 is allowed then set $\mathrm{COL}(z) = 2$.
3. If neither 1 or 2 is allowed then z cannot be colored. EXIT the loop and output $z - 1$ and $\mathrm{COL}(1)\mathrm{COL}(2) \cdots \mathrm{COL}(z-1)$. This indicates that the method was able to color $\{1, \ldots, z-1\}$ but no further.

See how far you get with the GREEDY algorithm.

What about if you have c colors? Then do the following modification:
$\mathrm{COL}(1) = 1$.
For $z = 2$ to ∞ (Assume $1, \ldots, z-1$ have been colored.)

1. Find the least $d \in \{1, \ldots, c\}$ that is allowed (or you may find it does not exist).
2. If d exists then set $\mathrm{COL}(z) = d$.
3. If no d exists (so no color is allowed) then z cannot be colored. EXIT the loop and output $z - 1$ and $\mathrm{COL}(1)\mathrm{COL}(2) \cdots \mathrm{COL}(z-1)$. This indicates that the method was able to color $\{1, \ldots, z-1\}$ but no further.

METHOD TWO: Greedy+Randomized

Intuition: when trying to color z, if there are many options pick one at random.

We will do the case of c colors.

COL(1) = 1.

For $z = 2$ to ∞ (Assume $1, \ldots, z-1$ have been colored.)

1. Find the SET D of allowed colors.
2. If $D \neq \emptyset$ then pick $d \in D$ at random and COL(z) = d.
3. If $D = \emptyset$ (so no color is allowed) then z cannot be colored. EXIT the loop and output $z-1$ and COL(1)COL(2) $\cdots$ COL($z-1$). This indicates that the method was able to color $\{1, \ldots, z-1\}$ but no further.

METHOD THREE: Backtracking

This is really FOUR algorithms.

It begins by doing either GREEDY or GREEDY+RAND

and it then does one of two things.

This is a method that you append to the end of either METHOD ONE or METHOD TWO.

We do this for c colors.

COL(1) = 1.

For $z = 2$ to ∞ (Assume $1, \ldots, z-1$ have been colored.)

1. Do the Greedy or Greedy+Randomized Algorithm. If it succeeds in coloring z then go to the next z.
2. (If you got here then z cannot be colored.) Hence there exists c pairs (x_1, y_1), …, (x_c, y_c) such that

   (a) For all i, $x_i < y_i$.

   (b) $y_1 < y_2 < \cdots < y_c$.

   (c) $x_1^2 + y_1^2 = z^2$ and $\mathrm{COL}(x_1) = \mathrm{COL}(y_1) = 1$.
   $x_2^2 + y_2^2 = z^2$ and $\mathrm{COL}(x_2) = \mathrm{COL}(y_2) = 2$.
   $\vdots$
   $x_c^2 + y_c^2 = z^2$ and $\mathrm{COL}(x_c) = \mathrm{COL}(y_c) = c$.

   We want to recolor one of $x_1, \ldots, x_c, y_1, \ldots, y_c$.

   (a) For $i = 1$ to c find the set of allowable colors D_i^x for x_i (not including i).

   (b) For $i = 1$ to c find the set of allowable colors D_i^y for y_i (not including i).

   (c) If $(\forall i)[D_i^x = \emptyset \wedge D_i^y = \emptyset]$ and (so no x_i or y_i can be recolor) then z cannot be colored. EXIT the loop and output $z-1$ and $\mathrm{COL}(1)\mathrm{COL}(2)\cdots\mathrm{COL}(z-1)$. This indicates that the method was able to color $\{1, \ldots, z-1\}$ but no further.

   (d) Pick a nonempty set among the D_i^x's and D_i^y's at random (or have some deterministic rule) Say its D_{10}^x. Then recolor x_{10} by some element of D_{10}^x picked WITH A RULE (its important this is NOT done at random) Once you do that CHECK if the coloring of $\{1, \ldots, z-1\}$ is still valid. If not then pick another color from D_{10}^x picked WITH A RULE. (We need a rule to make sure we try all options if need be.) If NONE of the colors in D_{10}^x work then choose another D_i^x or D_i^y at random (or have some determinstic rule) and try again.
   If you recolor (say) x_i then you can color z with i.

   (e) If NOTHING worked then EXIT the loop and output $z-1$ and $\mathrm{COL}(1)\mathrm{COL}(2)\cdots\mathrm{COL}(z-1)$. This indicates that the method was able to color $\{1, \ldots, z-1\}$ but no further.

NOTE- the algorithm above is not complete. I assumed that only one pair (x_i, y_i) prevented z from being colored i. Thats fine for now, but later fix it so that it deals with that case as well.

3 Approximation of Reals by Rationals

The talk and slides are on the question:

Is $\{a + b\sqrt{2} : a, b, \in \mathbb{Z}\}$ Dense?

However it leads into a project about approximating reals.

Slides:

`https://www.cs.umd.edu/~gasarch/COURSES/250/S24/slides/COMB/denseab.pdf`

Recording:

`https://www.cs.umd.edu/~gasarch/RAMSEYGANG/density.mp4`

Project Ideas

1. In the slides we proved the following towards the end (I changed the names of the variables because I need to for my next point).

   Theorem 3.1 *Let $\gamma \in \mathbb{I}$ (irrational). Then*

   $$(\exists^\infty q \in \mathbb{N})(\exists p \in \mathbb{Z})\left[\left|\gamma - \frac{p}{q}\right| < \frac{1}{q^2}.\right]$$

   Can we do better? For example, might the following be true:

   $$(\exists^\infty q \in \mathbb{N})(\exists p \in \mathbb{Z})[|\gamma - \frac{p}{q}| < \frac{1}{q^3}.$$

   Or perhaps a more modest goal:

   $$(\exists^\infty q \in \mathbb{N})(\exists p \in \mathbb{Z})[|\gamma - \frac{p}{q}| < \frac{1}{10q^2}.$$

   NO, this cannot be achieved. In the document.

   `https://www.cs.umd.edu/~gasarch/HURWITZ/hurwitz.pdf`

   I show (this was already known a long time ago) that for $\gamma = \frac{1+\sqrt{5}}{2}$

   $$(\exists N \in \mathbb{N})(\forall q \geq \mathbb{N})(\forall p \in \mathbb{N})[|\gamma - \frac{p}{q}| \geq \frac{1}{\sqrt{5}q^2}.$$

   In this project you will see how well irrationals can be approximated .

Details on this project on the next page.

4 TO DO List for Approx Irrationals Project

I will give three methods to generate a sequence of approximations to an irrational ζ. I will then say what you DO with the sequence, regardless of where it came from.

1) Continued Fractions Method READ up on continued fractions. WRITE a program that will, given an irrational ζ (you need to figure out how you can be GIVEN an irrational) generate the continued fraction expansion for ζ, up to (say) 100. Use this to generate a sequence of rational approximations for ζ

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_{100}}{q_{100}}.$$

2) Try lots of pairs $\{p, q\}$

Let $a \in \mathbb{N}$ such that $a < \zeta < a + 1$. We know that the only rationals worth considering are $\frac{p}{q}$ such that $a < \frac{p}{q} < a + 1$.

Write a program that looks at every $\{p, q\}$ with $1 \le p, q \le 1000$ where p, q are rel prime and $a < \frac{p}{q} < a + 1$. For each of these $\{p, q\}$ we have the rational $\frac{p}{q}$. These are our

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_{100}}{q_{100}}.$$

(There may be more or less than 100. If its to few then change 1000 to (say) 2000.)

3) Use Dirichlet's Proof

READ the following sides from slide 89 on.

`https://www.cs.umd.edu/~gasarch/COURSES/250/S24/slides/COMB/denseab.pdf`

The proof that you can get a good approx gives you a way to find it. Use this to get

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_{100}}{q_{100}}.$$

ONCE you have the sequence, what do to with it?

Assume you got the sequence

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_{100}}{q_{100}}$$

from one of the three methods above.

1. For $1 \le i \le 100$ find c_i such that $|\zeta - \frac{p_i}{q_i}| = \frac{c_i}{q_i^2}$.
2. Sort the $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_{100}}{q_{100}}$ based on the c_i. Put this in a table. Note if c_i's are inc, dec, or about the same.
3. If the c_i's are decreasing then try replacing q_i^2 with q_i^3.
4. Look at the data and see what you think.

5 Small Ramsey Numbers

Slides:

https://www.cs.umd.edu/~gasarch/COURSES/250/S24/slides/COMB/smallramseytalk.pdf

Recording:

https://www.cs.umd.edu/~gasarch/RAMSEYGANG/smallramsey.mp4

Project Ideas:

1. The proof that $R(a, b)$ exists actually gave you an algorithm to find the RED K_a or BLUE K_b: Look for a high RED degree or high BLUE degree and look at the RED neighbors or the BLUE neighbors. (And then you will need to recurse.) CODE THIS UP and try it out on random graphs.

2. Random Ramsey Numbers. We use $RR(4)$ as an example. It is know that every 2-coloring of the edges of K_{18} has a mono K_4. It is know that there exists a 2-coloring of the edges of K_{17} with no mono K_4. But what if you randomly color the edges of K_{17} a million times? Will you find a coloring with no mono K_4 (I know from last years projects that you will not.) Look at K_{16}, K_{15}, etc until you get (a) SOME coloring has a no mono K_4, (b) about half of them have no mono K_4, (c) other fractions. SAME project for K_3, K_5. Graph the fraction, so you can have a graph where as n goes from 4 to 18 see how the fraction of colorings that have a mono K_4 goes down.

   The above assumed that the random graphs have prob of RED is 1/2 and of BLUE is 1/2. You can vary that. What if RED is 3/4 and BLUE is 1/4. Then you may need to go much lower to find a 2-coloring with no mono K_4.

   You might want to use the algorithm in the first project to FIND the monochromatic graphs. I do not know if that will work since failure IS option.

3. Some of the colorings that had no large mono cliques came from number theory. This did not work for, say, $R(5)$. However, one could still look at graphs of that type and see if they have large mono cliques.

6 Ramsey Multiplicity

Slides:

https://www.cs.umd.edu/~gasarch/COURSES/752/S22/slides/2tritalk.pdf

Link to Recording and Passcode needed:
I have it twice- right now both work but just in case that changes
CLICK HERE FOR RECORDING
CLICK HERE FOR RECORDING
Passcode: YIj?2zbg

Project Ideas

1. Note the following contrast.

   (a) Computer Searches showed that every 2-coloring of the edges of K_{18} has 9 mono K_4's.

   (b) I showed using MATH that every 2-coloring of K_{19} has 2 mono K_4"s. By the first result we know this result is NOT the best possible.

   Try to give a MATH proof of better results. Perhaps a combination of MATH and mild Computer Work.

2. Try to prove an asymmetric theorem, something like For every 2-coloring of K_n there is either a mono K_3's or b mono K_4's. I will talk about some ideas on this when we meet. (This has already been done, but asymptotically. I would want to see some actual numbers.)

3. Use the proofs I gave to write a PROGRAM that will, given a 2-coloring of the edges of K_n, FIND the mono K_n's

4. Find the RANDOMIZED version of this problem. For example, for a random 2-coloring of K_n how many mono triangles do you get? I think its lots more than $\frac{n^3}{24}$.

7 The Probabilistic Method

The lecture first revisited UPPER bounds on $R(a, b)$ which is at the end of the Small Ramsey Numbers slides, and then proved LOWER bounds on $R(a, b)$ using The Prob Method.

The slides are here:

`https://www.cs.umd.edu/~gasarch/COURSES/752/S22/slides/probmethodtalk.pdf`

The recording was two lectures. The first is here:

`https://umd.zoom.us/rec/share/izZB3HF9olto2sHYDwpHb-NTveeFesjeLMFeWN4IR0GyXhA0Fo0VFDs7tu`
`6RGWP6YslHkeUgUj`

Passcode: 0#9PYWG$

The second recording is here:

`https://umd.zoom.us/rec/share/4DBYTNUx1DDokPvYl0O95xSoPAuZUv_6xVR9LZsCl-S8sew70KpM21cbfV`
`q1GvpAgOhi1pJYua`

Passcode: 6Vn*ga+K

Project Ideas Take *any* of the Theorems or Algorithms that used the prob method, and see what happens if you convert it to a real algorithm that flips coins. How well would it do?

TO DO

Dist Diff The numbers I put in (like $n = 1,000,000$) are guesses about what your computer can handle. Make such numbers easy-to-change.

1. Write a program that, given a set $A \subseteq \{1, \ldots, n\}$, determines if A has all distinct differences.

2. Write a program that will, given $n \in \mathbb{N}$ (say $n \geq 10$) and $0 < p < 1$, does the following: Form a set by, for each $i \in \{1, \ldots, n\}$, flip a biased coin (prob p of Y, probe $1 - p$ of N) to determine if i is in or out.

3. Write a program that will, given $n \in \mathbb{N}$ and $0 < p < 1$, form a set as in the last item, and then test if it is dist-diffs.

4. Write a program that will, given $n \in \mathbb{N}$ and $0 < p < 1$, form 100 sets and keeps track of how many of them are dist-diffs (you do not need to store the sets, just generate, test, and discard). Keep track of how what fraction of the sets are dist-diff.

5. For (say) $n = 1,000,000$ do the above for $p = \frac{1}{100}, \frac{1}{200}, \ldots, \frac{1}{10{,}000}$ and see when you get many diff-dist sets and when you get few.

6. Do the above for diff values of n. Note what is the largest p such that you get at least one diff-dist set.

7. Take all of this data and make conjectures of the form:

   For large n if you use $p =$ SOME FUNCTION OF n, you will get a diff-dist set of size BLAH.

8 Random Ramsey Project

Consider the following theorems:

1. For all 2-colorings of $\binom{[6]}{2}$ there is a mono K_3.
2. For all 2-colorings of $\binom{[6]}{2}$ there are two mono K_3.
3. For all 2-colorings of $\binom{[18]}{2}$ there is a mono K_4.
4. For all 2-colorings of $\binom{[18]}{2}$ there are four mono K_4.

The above theorems are true for *any* coloring. What about *most* colorings?

1. Write a program that will, k, n, and a 2-coloring of $\binom{[n]}{2}$, determines how many mono K_k's there are.
2. Write a program that will, given k, n, generate 10,000 random 2-colorings of $\binom{[n]}{2}$ and determine how many mono K_k's there are in each one. Find the MIN, MAX, AVG, and MEDIAN of that list of numbers.
3. Run the program with $k = 3$ and $n = 3, 4, 5, 6, 7, 8, 9, 10$ (as far as your computer can handle). For which n is the MIN always 1? For example, lets say its 4 (I DON"T KNOW IF IT IS). Then you have the following statement:

   For MOST 2-colorings of K_4 there exists a mono K_3.
4. Do the same for finding mono K_4's.
5. Do the same for number of mono K_3's and number of mono K_4's.
6. Recall that $R(5)$ is unknown. Find the smallest n such that for MOST 2-colorings of $\binom{[n]}{2}$ there is a mono K_5. Try to go higher!
7. Later you will learn VDW's theorem and Rado's Theorem and do similar studies of them.

9 Ramsey Theory and the Primes

This website

`https://www.cs.umd.edu/~gasarch/TOPICS/ramseyprimes/ramseyprimes.html`
has the following.

1. Links to FIVE papers that prove primes are infinite from Ramsey Theory.
2. My slides from my talk that had my paper on this topic, plus more on the primes.

Unfortunately there is no recording of that lecture.

TO DO list for the Ramsey-Primes-Gang. If you do EVERYTHING on this list then you have finished the project, though we can then look at other things.

1. Read my slides to make sure you understand them.
2. Read my paper on Ramsey Theory and Primes- it will have some more things in it that were not in the talk.
3. Read the other papers, just the parts where they proof the primes are infinite. You may have questions for me since they use some things in Ramsey Theory that you do not know yet. Feel free to email me, or ask me next Tuesday. I will be doing Van Der Waerden's theorem next week so that might be all you need. For now you might just need the STATEMENT of it.
4. Write their proofs that primes are infinite in your own words.
5. See where their proof goes wrong for $\mathbb{Q}$ and for $\mathbb{Q}_2$. (I suspect that for $\mathbb{AI}$ there proof goes wrong since $\mathbb{AI}$ is not atomic.