A Clean CFG and Proof For
$\{w : \#_b(w) = m\#_a(w)\}$
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ROB - this is part you helped me with. I state it as one lemma, but it will be part of a much bigger lemma that covers all of the cases.

$k \ge m+1$ and $\ell \ge m+1$. (THIS WAS THE HARD CASE)

Lemma 0.1 *Let $m \in \mathsf{N}$. Let*

$$L = \{w : \#_b(w) = m\#_a(w) + 0\}.$$

Let $w \in L$. Let $w = w_1 \cdots w_{(m+1)n}$. (There are n $a's$ and mn b's.) If $w = b^k a w' a b^\ell$ where $k, \ell \ge k+1$ then one of the following occurs.

1. *There exists $x, y \in L$ such that $w = xy$.*
2. *There exists $x, y \in L$ such that*

Proof:

Notation 0.2 Let $x \in \{a,b\}^*$.

1. $\#_a(x)$ is the number of a's in x.
2. $\#_b(x)$ is the number of b's in x.
3. $\text{weight}(x) = \#_a(x) - \frac{\#_b(x)}{m}$.

Note that

$$\text{weight}(b^k a) = 1 - \frac{k}{m} < 0.$$

Note that

$$\text{weight}(b^k a w') = (\#_a(w) - \#_a(ab^\ell)) - \frac{1}{m}(\#_b(w) - \#_b(ab^\ell)) = (n-1) - \frac{1}{m}(mn - \ell) = -1 + \frac{\ell}{m} > 0$$

Hence there must be a prefix of w of the form $b^k a z'$ where the weight is ≥ 0. Consider the shortest such extension. It must end in a, so let it be $b^k a z a$.

Case 1 weight$(b^k aza) = 0$. Then let $x = b^k aza$ and y be the rest of the string. Clearly $x, y \in L$.

Case 2 weight$(b^k aza) > 0$. Since the last a pushed the weight from positive to negative we must have the following:

$$\text{weight}(b^k az) = -\frac{1}{m}$$

So

$$\#_a(b^k az) - \frac{\#_b(b^k az)}{m} = -\frac{1}{m}$$

$$\#_a(az) - \frac{k + \#_b(az)}{m} = -\frac{1}{m}$$

$$\#_a(az) = \frac{k - 1 + \#_b(az)}{m}$$

$$m\#_a(az) = k - 1 + \#_b(az)$$

$$\#_b(az) = m\#_a(az) + 1 - k$$

$$\#_b(b^{k-1}az) = k-1+\#_b(az) = k-1+m\#_a(az)+1-k = m\#_a(az) = m\#_a(b^{k-1}z).$$

So $b^{k-1}az \in L$. Hence w has a prefix of the form bxa where $x \in L$.
By the same reasoning, w has a suffix of the form ayb where $y \in L$.

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