Relations and Functions

Problem: Let $A$ and $B$ be arbitrary sets. How many different relations are there from a set $A$ to a set $B$?

Solution: Note that all such relations are subsets of the set $A \times B$. In other words, the question is equivalent to asking the question, how many subsets are there of the set $A \times B$.

Recall that $2^{(A \times B)}$ is the set of all subsets of $A \times B$. The cardinality of the powerset is

$$|2^{A \times B}| = 2^{|A \times B|} = 2^{|A| \times |B|}$$

Recall the following properties on relations:
Let $R$ be a relation defined on set $A$. We say that $R$ is

- **reflexive**, if for all $x \in A$, $(x, x) \in R$.
- **irreflexive**, if for all $x \in A$, $(x, x) \notin R$.
- **symmetric**, if for all $x, y \in A$, $(x, y) \in R \implies (y, x) \in R$.
- **antisymmetric**, if for all $x, y \in A$, $x R y$ and $y R x \implies x = y$.
- **transitive**, if for all $x, y, z \in A$, $x R y$ and $y R z \implies x R z$.

**Problem:** What are the properties of the following relations?

- $R_1$ : “is a sibling of” relation on the set of all people.
- $R_2$ : “$\leq$” relation on $\mathbb{Z}$.
- $R_3$ : “$<$” relation on $\mathbb{Z}$.
- $R_4$ : “$|$” relation on $\mathbb{Z}^+$.
- $R_5$ : “$|$” relation on $\mathbb{Z}$. 
Solution.

Reflexive: $R_2, R_4$
Irreflexive: $R_1, R_3$
Symmetric: $R_1$
Antisymmetric: $R_2, R_3, R_4$
Transitive: $R_2, R_3, R_4, R_5$

Note that $R_5$ is not reflexive because $(0,0) \notin R_5$; it is not antisymmetric because for any integer $a$, $a|\neg a$ and $\neg a|a$, but $a \neq -a$. Observe that $R_5$ is an example of a relation that is neither symmetric nor antisymmetric.

Equivalence Relations

A relation $R$ on a set $A$ is an equivalence relation if and only if it is reflexive, symmetric and transitive.

Prove: Let $A$ be the set of all strings of English letters. Suppose that $R$ is the relation on the set $A$ such that $a R b$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string $x$. Prove that $R$ is an equivalence relation.
**Solution:** To show that $R$ is an equivalence relation, we need to prove that $R$ is reflexive, symmetric, and transitive.

- **Reflexive:** Let $a$ be an arbitrary string in $A$. Note that $l(a) = l(a)$, and hence $a R a$. This shows that $R$ is reflexive.

- **Symmetric:** Let $a, b$ be arbitrary elements in $A$. Assume $(a, b) \in R$. Since $a R b$, this means that $l(a) = l(b)$. Hence $l(b) = l(a)$, so $b R a$. This shows that $R$ is symmetric.

- **Transitive:** Let $a, b, c$ be arbitrary elements in $A$. Assume that $(a, b), (b, c) \in R$. Thus $l(a) = l(b)$ and $l(b) = l(c)$, which implies that $l(a) = l(c)$. Hence $a R c$ and $R$ is transitive.

Since $R$ is reflexive, symmetric, and transitive, it is an equivalence relation.
Operations on Relations

Since relations are sets, we can take a relation or a pair of relations and produce a new relation using set operations.

Examples:

- Let “>” be the greater than relation on the set of integers. Let “<” be the less than relation on the set of integers.
  
    Then “>” ∪ “<” = “≠”

- Let “≥” be the greater than or equal relation on the set of integers. Let “=” be the equal relation on the set of integers.
  
    Then “≥” \ “=” = “>”.

Functions

Let $A$ and $B$ be sets. A function from $A$ to $B$ is a relation, $f$, from $A$ to $B$ such that for all $a \in A$ there is
exactly one $b \in B$ such that $(a, b) \in f$.

Here are some definitions:

- If $(a, b) \in f$, then we write $b = f(a)$.

- A function from $A$ to $B$ is also called a **mapping** from $A$ to $B$ and we write it as $f : A \to B$.

- The set $A$ is called the **domain** of $f$ and the set $B$ the **codomain**.

- If $a \in A$ then the element $b = f(a)$ is called the **image** of $a$ under $f$. The **range** of $f$, denoted by $\text{Ran}(f)$ is the set

$$\text{Ran}(f) = \{b \in B \mid \exists a \in A \text{ s.t. } b = f(a)\}$$

- Two functions are **equal** if they have the same domain, have the same codomain, and map each element of the domain to the same element in the codomain.

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**Examples:**
• Some functions are ones that a familiar to ones that you may have studied before. For example: $f_1 : \mathbb{Z} \to \mathbb{Z}, f_1(x) = x^2$

• Functions need not have such a clean definition. For example:

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then there can be a function $f_2 : A \to B$, such that $f_2(1) = b$, $f_2(2) = a$, $f_2(3) = b$.

Let $A$ and $B$ be sets. Let $f : A \to B$ be a function.

• $f$ is said to be **injective**, iff $\forall x, y \in A, x \neq y \implies f(x) \neq f(y)$.

Sometimes it is informative to look at its contra-positive statement:

$\forall x, y \in A, f(x) = f(y) \implies x = y$.

• $f$ is called **surjective**, iff $\forall b \in B, \exists a \in A, f(a) = b$.

• $f$ is a **bijection**, iff it is both surjective and injective.
**Prove:** Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$, such that $f(x) = x + 1$. Prove that $f$ is bijective.

To prove that $f$ is bijective, we wish to show that it is injective and surjective.

- **Injective:** Let $x$ and $y$ be arbitrary elements in $A$. Assume that $x \neq y$. Then $f(x) = x + 1 \neq y + 1 = f(y)$. Since $f(x) \neq f(y)$, then we have shown that $f$ is injective.

- **Surjective:** Let $x$ be an arbitrary element in $B$. Let $y = x - 1$. Note that $f(y) = f(x - 1) = x$. Hence, since there is a $y \in \mathbb{Z}$ such that $f(y) = x$, we have that $f$ is surjective.

Since we have shown that the function is injective and surjective, we have that it is bijective.

**Injection and Surjection Rule**

**The Injection Rule**
Let $A$ and $B$ be two finite sets. If there is an injective function from $A$ to $B$, then $|A| \leq |B|$.

We can see this as follows. Since each element in $A$ is mapped to a distinct element in $B$, this means that $|A| = |\text{Ran}(f)|$. Further, since $\text{Ran}(f) \subseteq B$, we know that $|\text{Ran}(f)| \leq |B|$. Therefore, $|A| \leq |B|$.

**The Surjection Rule**

Let $A$ and $B$ be two finite sets. If there is an surjective function from $A$ to $B$, then $|A| \geq |B|$.

We can see this as follows. Suppose for the sake of contradiction that there is a surjective function, but $|A| < |B|$. Note that since each element in $A$ is mapped to exactly one element in $B$, it must be that $|A| \geq |\text{Ran}(f)|$. Since $|\text{Ran}(f)| \leq |A|$ and $|A| < |B|$, we have that $|\text{Ran}(f)| < |B|$. Since $\text{Ran}(f) \subseteq B$ and $|\text{Ran}(f)| < |B|$, it must be that $\text{Ran}(f) \subset B$. Therefore, we have that $B \setminus \text{Ran}(f) \neq \emptyset$. In other words, there is an element in $B$ such that it is not mapped onto by the function $f$. This contradicts the assumption that $f$ is surjective.