Relations and Functions

Problem: Let $A$ and $B$ be arbitrary sets. How many different relations are there from a set $A$ to a set $B$?

Solution: Note that all such relations are subsets of the set $A \times B$. In other words, the question is equivalent to asking the question, how many subsets are there of the set $A \times B$.

Recall that $2^{(A \times B)}$ is the set of all subsets of $A \times B$. The cardinality of the powerset is

$$|2^{A \times B}| = 2^{|A \times B|} = 2^{|A| \times |B|}$$

Recall the following properties on relations:

Let $R$ be a relation defined on set $A$. We say that $R$ is

- **reflexive**, if for all $x \in A$, $(x, x) \in R$.
- **irreflexive**, if for all $x \in A$, $(x, x) \notin R$.
- **symmetric**, if for all $x, y \in A$, $(x, y) \in R \implies (y, x) \in R$.
- **antisymmetric**, if for all $x, y \in A$, $x R y$ and $y R x \implies x = y$.
- **transitive**, if for all $x, y, z \in A$, $x R y$ and $y R z \implies x R z$.

Problem: What are the properties of the following relations?

- $R_1$: “is a sibling of” relation on the set of all people.
- $R_2$: “$\leq$” relation on $\mathbb{Z}$.
- $R_3$: “$<$” relation on $\mathbb{Z}$.
- $R_4$: “$|$” relation on $\mathbb{Z}^+$. 
- $R_5$: “$|$” relation on $\mathbb{Z}$.

Solution.

- Reflexive: $R_2, R_4$
- Irreflexive: $R_1, R_3$
- Symmetric: $R_1$
- Antisymmetric: $R_2, R_3, R_4$
- Transitive: $R_2, R_3, R_4, R_5$
Note that $R_5$ is not reflexive because $(0, 0) \not\in R_5$; it is not antisymmetric because for any integer $a$, $a | -a$ and $-a | a$, but $a \neq -a$. Observe that $R_5$ is an example of a relation that is neither symmetric nor antisymmetric.

**Equivalence Relations**

A relation $R$ on a set $A$ is an *equivalence relation* if and only if it is reflexive, symmetric and transitive.

**Prove:** Let $A$ be the set of all strings of English letters. Suppose that $R$ is the relation on the set $A$ such that $a R b$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string $x$. Prove that $R$ is an equivalence relation.

**Solution:** To show that $R$ is an equivalence relation, we need to prove that $R$ is reflexive, symmetric, and transitive.

- **Reflexive:** Let $a$ be an arbitrary string in $A$. Note that $l(a) = l(a)$, and hence $a R a$. This shows that $R$ is reflexive.

- **Symmetric:** Let $a, b$ be arbitrary elements in $A$. Assume $(a, b) \in R$. Since $a R b$, this means that $l(a) = l(b)$. Hence $l(b) = l(a)$, so $b R a$. This shows that $R$ is symmetric.

- **Transitive:** Let $a, b, c$ be arbitrary elements in $A$. Assume that $(a, b), (b, c) \in R$. Thus $l(a) = l(b)$ and $l(b) = l(c)$, which implies that $l(a) = l(c)$. Hence $a R c$ and $R$ is transitive.

Since $R$ is reflexive, symmetric, and transitive, it is an equivalence relation.

**Operations on Relations**

Since relations are sets, we can take a relation or a pair of relations and produce a new relation using set operations.

**Examples:**

- Let “$>$” be the greater than relation on the set of integers. Let “$<$” be the less than relation on the set of integers.

  Then “$>$” $\cup$ “$<$” = “$\neq$”

- Let “$\geq$” be the greater than or equal relation on the set of integers. Let “$=$” be the equal relation on the set of integers.

  Then “$\geq$” $\setminus$ “$=$” = “$>$”.
Functions

Let $A$ and $B$ be sets. A function from $A$ to $B$ is a relation, $f$, from $A$ to $B$ such that for all $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$.

Here are some definitions:

- If $(a, b) \in f$, then we write $b = f(a)$.
- A function from $A$ to $B$ is also called a mapping from $A$ to $B$ and we write it as $f : A \rightarrow B$.
- The set $A$ is called the domain of $f$ and the set $B$ the codomain.
- If $a \in A$ then the element $b = f(a)$ is called the image of $a$ under $f$. The range of $f$, denoted by $\text{Ran}(f)$ is the set
  \[
  \text{Ran}(f) = \{b \in B \mid \exists a \in A \text{ s.t. } b = f(a)\}
  \]
- Two functions are equal if they have the same domain, have the same codomain, and map each element of the domain to the same element in the codomain.

Examples:

- Some functions are ones that a familiar to ones that you may have studied before. For example: $f_1 : \mathbb{Z} \rightarrow \mathbb{Z}, f_1(x) = x^2$
- Functions need not have such a clean definition. For example:

  Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then there can be a function $f_2 : A \rightarrow B$, such that $f_2(1) = b$, $f_2(2) = a$, $f_2(3) = b$.

Let $A$ and $B$ be sets. Let $f : A \rightarrow B$ be a function.

- $f$ is said to be **injective**, iff $\forall x, y \in A, x \neq y \implies f(x) \neq f(y)$.
  Sometimes it is informative to look at its contrapositive statement:
  $\forall x, y \in A, f(x) = f(y) \implies x = y$.
- $f$ is called **surjective**, iff $\forall b \in B, \exists a \in A, f(a) = b$.
- $f$ is a **bijection**, iff it is both surjective and injective.

**Prove:** Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$, such that $f(x) = x + 1$. Prove that $f$ is bijective.

To prove that $f$ is bijective, we wish to show that it is injective and surjective.

- **Injective:** Let $x$ and $y$ be arbitrary elements in $A$. Assume that $x \neq y$. Then $f(x) = x + 1 \neq y + 1 = f(y)$. Since $f(x) \neq f(y)$, then we have shown that $f$ is injective.
- **Surjective:** Let \( x \) be an arbitrary element in \( B \). Let \( y = x - 1 \). Note that \( f(y) = f(x - 1) = x \). Hence, since there is a \( y \in \mathbb{Z} \) such that \( f(y) = x \), we have that \( f \) is surjective.

Since we have shown that the function is injective and surjective, we have that it is bijective.

**Injection and Surjection Rule**

**The Injection Rule**

Let \( A \) and \( B \) be two finite sets. If there is an injective function from \( A \) to \( B \), then \( |A| \leq |B| \).

We can see this as follows. Since each element in \( A \) is mapped to a distinct element in \( B \), this means that \( |A| = |\text{Ran}(f)| \). Further, since \( \text{Ran}(f) \subseteq B \), we know that \( |\text{Ran}(f)| \leq |B| \). Therefore, \( |A| \leq |B| \).

**The Surjection Rule**

Let \( A \) and \( B \) be two finite sets. If there is a surjective function from \( A \) to \( B \), then \( |A| \geq |B| \).

We can see this as follows. Suppose for the sake of contradiction that there is a surjective function, but \( |A| < |B| \). Note that since each element in \( A \) is mapped to exactly one element in \( B \), it must be that \( |A| \geq |\text{Ran}(f)| \). Since \( |\text{Ran}(f)| \leq |A| \) and \( |A| < |B| \), we have that \( |\text{Ran}(f)| < |B| \). Since \( \text{Ran}(f) \subseteq B \) and \( |\text{Ran}(f)| < |B| \), it must be that \( \text{Ran}(f) \subset B \). Therefore, we have that \( B \setminus \text{Ran}(f) \neq \emptyset \). In other words, there is an element in \( B \) such that it is not mapped onto by the function \( f \). This contradicts the assumption that \( f \) is surjective.