Principle of Inclusion-Exclusion

Problem: How many strings are there of four lower-case letters that have the letter x in them?

Solution: Let $S$ be the set of all possible four-letter strings that can be constructed using lower-case letters. The set $S$ can be partitioned into two sets $S_1$ and $S_2$ where $S_1$ is the set of all strings that contain at least one x and $S_2$ is the set of strings that do not contain x. Hence we have

$$|S| = |S_1| + |S_2|$$  (1)

Let us determine $|S|$, and let the outcome in $S$ be a 4-tuple that directly represents the string. We can construct an outcome in the following way:
Step 1. Choose the 1st letter – 26 ways
Step 2. Choose the 2nd letter – 26 ways
Step 3. Choose the 3rd letter – 26 ways
Step 4. Choose the 4th letter – 26 ways

By the multiplication rule, \(|S| = 26^4\).

Let us determine \(|S_2|\), and let the outcome in \(S_2\) be a 4-tuple that directly represents the string. We can construct an outcome in the following way:

Step 1. Choose the 1st letter – 25 ways (any letter but x)
Step 2. Choose the 2nd letter – 25 ways
Step 3. Choose the 3rd letter – 25 ways
Step 4. Choose the 4th letter – 25 ways

By the multiplication rule, \(|S_2| = 25^4\).

Substituting these values in equation (1) we get

\[ |S_1| = 26^4 - 25^4 = 66351 \]
Incorrect Solution. Here is an incorrect solution. Can you figure out what is wrong?

A four letter string that contains $x$ can be constructed in two steps as follows.

- Choose a letter to be $x$ – 4 ways
- Choose the other 3 letters – $26^3$ ways

By the multiplication rule, there are $4 \cdot 26^3 = 70304$ four letter strings that contain $x$.

Problem:

A certain course consists of 75 people, with an equal number of people from each class (25 freshman, 25 sophomores, and 25 juniors). The professor wants to form a committee of 9 people such that the committee contains at least one person from each year. How many ways can the professor form such a committee?

Solution:

We solve the problem by using complementary counting. First, we find the total number of ways to form committees with no restriction. This is simply choosing 9 people
from 75, or \( \binom{75}{9} \).

Next, we need to subtract the number of committees that are missing at least one of the years. We use the following sets:

- \( C_1 \): Committees that don’t have any freshman
- \( C_2 \): Committees that don’t have any sophomores
- \( C_3 \): Committees that don’t have any juniors

Note that \( C_1 \cup C_2 \cup C_3 \) contains all of the committees that are missing at least one of the class years. In order to determine \( |C_1 \cup C_2 \cup C_3| \), we can use PIE.

\[
|C_1 \cup C_2 \cup C_3| = |C_1| + |C_2| + |C_3| - |C_1 \cap C_2| - |C_2 \cap C_3| - |C_1 \cap C_3|
\]

Let us calculate the cardinalities of each of the intersections:

- \( |C_1|, \ |C_2|, \ |C_3| \)

The cardinality of each of these is equal to \( \binom{50}{9} \) since we just choose 9 people from the 50 people that aren’t a part of the missing year.
• $|C_1 \cap C_2|$, $|C_2 \cap C_3|$, $|C_1 \cap C_3|$

The intersection of two of these sets is the set containing all committees that are missing two class years. The cardinality of each of these is equal to $\binom{25}{9}$ since we just choose 9 people from the 25 people of the remaining year.

• $|C_1 \cap C_2 \cap C_3|$

The intersection of three of these sets is the set containing all committees that are missing three class years. Clearly this is equal to 0 since we can’t form a committee with 0 people.

Applying the inclusion-exclusion principle, we get:

$$|C_1 \cup C_2 \cup C_3| = 3 \binom{50}{9} - 3 \binom{3}{2} \binom{25}{9} + 0$$

Thus, there are

$$\frac{75}{9} - 3 \frac{50}{9} - 3 \frac{25}{9}$$

committees that the professor can form.
Permutations of Multisets

Let $S$ be a multiset that consists of $n$ objects of which

- $n_1$ are of type 1 and indistinguishable from each other.
- $n_2$ are of type 2 and indistinguishable from each other.
- $\vdots$
- $n_k$ are of type $k$ and indistinguishable from each other.

and suppose $n_1 + n_2 + \ldots + n_k = n$. What is the number of distinct permutations of the $n$ objects in $S$?

A permutation of $S$ can be constructed by the following $k$-step process:

1. Choose $n_1$ places out of $n$ places for type 1 objects.
2. Choose $n_2$ places out of the remaining $n - n_1$ places for type 2 objects.
Step k. Choose \( n_k \) places of the remaining unused places for type \( k \) objects.

By the multiplication rule, the total number of permutations of \( n \) objects in \( S \) is

\[
\binom{n}{n_1} \binom{n-n_1}{n_2} \cdots \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}
\]

\[
= \frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{n-n_1-n_2-\cdots-n_{k-1}}{n_k!(n-n_1-\cdots-n_k)!}
\]

\[
= \frac{n!}{n_1!n_2!\cdots n_k!}
\]

**Example.** How many permutations are there of the word MISSISSIPPI?

**Solution.** We want to find the number of permutations of the multiset \( \{1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P\} \). Thus, \( n = 11, n_1 = 1, n_2 = 4, n_3 = 4, n_4 = 2 \). Then number of permutations is given by

\[
\frac{n!}{n_1!n_2!n_3!n_4!} = \frac{11!}{1!4!4!2!}
\]