CMSC 250 Assigned: July 11, 2017

Homework 9

Due: July 14, 2017

This assignment is due at 12PM on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified.

Each solution must be written independently by yourself - no collaboration is allowed.

In your answers, you may use results that we have proved in lecture, results from discussion sessions, and results from previous homeworks as building blocks for your solutions. We also allow you to assume the results for the parity of sums and products (i.e. you can assume odd + odd = even, even × odd = even, etc.) Everything else you will have to prove on your own!

Remember that your solutions must be typeset in LaTeX. You can find a template for this homework on the course site. Make sure that each problem (by question number) is separated from each other (i.e. you should not have the solutions to Question 1 and Question 2 on the same page).

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1. [10 pts] Krishna's bakery has two dedicated customers (Matt and Shawn) that, every morning, buy a assortment of a dozen baked goods each: each pastry Matt buys is a cupcake with probability $\frac{2}{3}$ and a donut with probability $\frac{1}{3}$, while each pastry Shawn buys is a donut with probability $\frac{2}{3}$ and a cupcake with probability $\frac{1}{3}$. Matt and Shawn arrive at Krishna's bakery right when it opens (one of them is always the first customer at the register). Once a customer reaches the register, they choose one item at a time to buy with the aforementioned probabilities, until they have purchased their dozen (whoever comes second unfortunately has to wait for him to finish). Given that they are equally likely to arrive at the register first:

(a) Show that the probability of a donut being the first baked good bought on any given day is $\frac{1}{2}$.

(b) If the first two baked goods bought are cupcakes, what is the probability the third baked good bought is a donut?

(c) If the first two baked goods bought are donuts, what is the probability that Matt was first to the register?

2. [10 pts] Let G be a graph with n vertices and exactly $n-1$ edges. Prove that G has either a vertex of degree 1 or an isolated vertex.

3. [10 pts] Prove that a connected graph G with s vertices has exactly one cycle if and only if there are a total of s edges.

4. [10 pts] Let T be a tree with $n > 1$ vertices. Show that the number of leaves is

$$2 + \sum_{v_i \in V : \deg(v_i) \geq 3} (\deg(v_i) - 2)$$

5. [10 pts] After a bountiful Thanksgiving (yes, pirates celebrate Thanksgiving too), Sparrow is feeling particular charitable and wants the RubyPerl to do some "good" for once. Sparrow decides that his crew will steal from the rich and give to the poor (to emulate one of his childhood heroes). He decides to roll a die whenever he sees a ship. If it isn't a six, he robs the ship and lays in wait for the next ship. If it is a six, he will pillage the ship anyways (because why would he waste a perfectly good ship) but then he will give his crew the rest of the day off. However, since his pirates all ate too much during Thanksgiving dinner and are tired, he vows to them that they will pillage at most four ships. Let X be the number of ships they pillage. Find $\mathbf{E}[X]$.

6. [10 pts] After having proven himself at every turn and demonstrated his excellence as a pirate, Sparrow decides, that his ship should be as iconic as he is. With this in mind, he elects to carve letters into the hull of the Ruby Perl. After discussing his idea with his most trusted underlings, he decides that writing "RUBYPERL" on the side of the ship would be too basic for a pirate of his stature, and instead they should write letters randomly going all the way around the ship.

Sparrow still wants to be able to find the word "RUBYPERL" carved into the ship, so he asks you: how many letters would he need to write randomly to have expected number of occurrences of the string "RUBYPERL" be 1?

(Assuming that all letters 26 letters of the alphabet are used with equal probability and everything is capital because this is a pirate ship of course).