CMSC 250 Assigned: June 30, 2017

# Homework 8

Due: July 3, 2017

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This assignment is due at 12PM on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified.

Each solution must be written independently by yourself - no collaboration is allowed.

In your answers, you may use results that we have proved in lecture, results from discussion sessions, and results from previous homeworks as building blocks for your solutions. We also allow you to assume the results for the parity of sums and products (i.e. you can assume odd + odd = even, even × odd = even, etc.) Everything else you will have to prove on your own!

Remember that your solutions must be typeset in LaTeX. You can find a template for this homework on the course site. Make sure that each problem (by question number) is separated from each other (i.e. you should not have the solutions to Question 1 and Question 2 on the same page).

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**1. [10 pts]** There are $n$ Mega Seeds in a box, where $n \geq 2$. We repeatedly do the following until each box has just one Mega Seed in it: Pick a box containing $m > 1$ Mega Seeds and split it into two boxes by taking some $s$ seeds to place into a new box, where $1 \leq s < m$. Say that there are now $r$ seeds in the original box. When we do this, we also compute the product $rs$.

Prove, using induction, that the sum of all the products is $\binom{n}{2}$ no matter how we split up the boxes in each turn.

For example, if we start with 4 Mega Seeds, we can split this up in the following way:

1. Split the 4 Mega Seeds into a box of 3 seeds and a box of 1 seed.

2. Split the box of 3 Mega Seeds into a box of 2 seeds and a box of 1 seed.

3. Split the box of 2 seeds into two boxes of 1 seed

The product in the first step is $3 \times 1 = 3$, the second is $2 \times 1 = 2$, and the last step is $1 \times 1 = 1$. The sum of these is $3 + 2 + 1 = 6 = \binom{4}{2}$.

Alternately, the following steps could have been performed:

1. Split the 4 Mega Seeds into two boxes of 2 seeds.

2. Split the first box of 2 Mega Seeds into two boxes of 1 seed

3. Split the second box of 2 seeds into two boxes of 1 seed

The product in the first step is $2 \times 2 = 4$, the second is $1 \times 1 = 1$, and the last step is $1 \times 1 = 1$. The sum of these is $4 + 1 + 1 = 6 = \binom{4}{2}$.

**2. [10 pts]** Consider $n$ arbitrary integers.

Prove, using induction, that the sum of the integers is odd if and only if there is an odd number of odd integers.

Assume that $n \in \mathbb{Z}^+$. You must invoke your Induction Hypothesis in both directions of your proof in your Induction Step.

**3. [10 pts]** Suppose that Kyle, Kenny, Cartman, and Stan have 210 otters each. In order to keep track of them all, each one numbers their otters from 1 to 210, and each otter receives a collar with its number on it. Unfortunately, otter collars do not come in many varieties, so all of Kyle and Kenny's otters have green collars, and all of Cartman and Stan's otters have yellow collars. The otters are released into two large pools. So that the owners can tell them apart, Kyle and Cartman's otters are in one pool and Kenny and Stan's otters are in the other. We now pick one otter from each pool independently and uniformly at random:

(a) Given that at least one of the otters picked has a green collar, what is the probability that both otters chosen have green collars?

(b) Given that at least one of the otters has a green collar and is numbered 13, what is the probability that both otters have a green collar?

**4. [12 pts]** Consider a probabilistic game which involves tossing 6 fair coins in succession. Let us define the following events in the resulting probabilistic space:

$A$: event that the first toss results in a Tails.

$B$: event that the third toss results in a Heads.

$C$: event that there are a total of 3 Heads.

$D$: event that the longest repeated sequence

(sequence of repeated Heads or Tails) is at least length 2.

$E$: event that the longest repeated sequence (sequence of repeated Heads or Tails) is at most length 5.

Answer the following questions giving proper justification. You should show that the events are independent by showing that, for two events $X$ and $Y$, that:

$$\Pr[X \cap Y] = \Pr[X] \cdot \Pr[Y]$$

For 3 events $X$, $Y$, $Z$, you need to show that the events are pairwise independent, and that:

$$\Pr[X \cap Y \cap Z] = \Pr[X] \cdot \Pr[Y] \cdot \Pr[Z]$$

(a) Are events $A$ and $B$ independent?

(b) Are events $A$ and $C$ independent?

(c) Are events $B$ and $C$ independent?

(d) Are events $A, B$, and $C$ independent?

(e) Are events $D$ and $E$ independent?

**5. [10 pts]** Sparrow is upset. The LWPSA (Large Wooden Pirate Ship Association) recently passed some new regulations. To ensure that no two pirates are terrorizing the sea at the same time, all pirate ships are required to register ahead of time exactly when they will be patrolling the seas. Sparrow gets frustrated by the new regulations and sends Will to handle registration in his place.

Will sees there are 30 different possible times to register for (none of these times conflict with each other). Specifically, each day of the week (Monday to Friday) has 6 available timeslots. Will doesn't want the crew to be too overworked, so he decides to select 7 out of the 30 slots uniformly at random, and register for those.

If all timeslots are equally likely to be chosen (and none of them overlap), what is the probability that the Ruby Perl (Sparrow's ship) will end up patrolling the seas at least once each day from Monday through Friday (inclusive).