CMSC 250 Assigned: June 27, 2017

# Homework 7

Due: June 30, 2017

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This assignment is due at 12PM on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified.

Each solution must be written independently by yourself - no collaboration is allowed.

In your answers, you may use results that we have proved in lecture, results from discussion sessions, and results from previous homeworks as building blocks for your solutions. We also allow you to assume the results for the parity of sums and products (i.e. you can assume odd + odd = even, even $\times$ odd = even, etc.) Everything else you will have to prove on your own!

Remember that your solutions must be typeset in LaTeX. You can find a template for this homework on the course site. Make sure that each problem (by question number) is separated from each other (i.e. you should not have the solutions to Question 1 and Question 2 on the same page).

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**1. [12 pts]**

(a) Prove that $\forall n \in \mathbb{N}$, $3^{3n+1} + 2^{n+1}$ is divisible by 5.

(b) Prove the following using induction. For all positive integers $n$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

**2. [10 pts]** Let $F_n$ denote the $n^{th}$ Fibonacci number. Prove that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, for all $n \geq 2$.

Note: the Fibonacci numbers are defined as $F_1 = F_2 = 1$, $F_i = F_{i-1} + F_{i-2}$ for $i \geq 3$.

**3. [10 pts]** Consider a binary string consisting of $n$ bits, where $n \geq 1$. We are allowed to perform the following operation: we can replace a `1` by a `x`, and when we do that we must flip the two bits immediately adjacent to it that remain. Thus `1100` becomes `0x10` when we replace the `1` in the middle by a `x`, and then we get `0xx1` when we replace the new `1`. Prove that we can convert a binary string of length $n$ into a string of $n$ `x`'s (we refer to this as "crossing out" a string) using the above operation if and only if the string has an odd number of `1`s.

**4. [10 pts]** Consider the following game. Initially there are two tokens in a row; the left one is blue and the right one is red. You can change the configuration by performing a number of moves. In each move, you can either insert two successive tokens of the same color (red or blue) or remove two successive tokens of the same color. Note that you can insert into or remove from any position that you wish.

(a) Let $r_e$ be the number of red tokens at even positions, and $r_o$ be the numbe of red tokens at odd positions. Prove, using induction, that $r_e - r_o = 1$ after any $n$ moves.

(b) Is it possible to produce a configuration where there are exactly two tokens, the left token being red, and the right one being blue?

**5. [10 pts]** Consider a $3 \times 3$ grid composed of 9 unit squares. Each of the unit squares is randomly colored black or white. What is the probability that at least one $2\times2$ square in this grid will be completely black?

**6. [12 pts]** Suppose we roll 3 fair 20-sided dice, such that each die takes on values between 1-20 independent of each other, and uniformly at random.

(a) What is the probability that they all have distinct values?

(b) What is the probability that they do not all have the same value (so two dice can have the same value, but not all three)?

(c) What is the probability that the sum of the dice rolls is even?

(d) What is the probability that all the values on the dice are even, and that they sum to 24?

(e) What is the probability that the sum of the values of the dice is less than or equal to 23?