CMSC 250 Assigned: June 23, 2017

Homework 6

Due: June 27, 2017

This assignment is due at 12PM on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified.

Each solution must be written independently by yourself - no collaboration is allowed.

In your answers, you may use results that we have proved in lecture, results from discussion sessions, and results from previous homeworks as building blocks for your solutions. We also allow you to assume the results for the parity of sums and products (i.e. you can assume odd + odd = even, even × odd = even, etc.) Everything else you will have to prove on your own!

Remember that your solutions must be typeset in LaTeX. You can find a template for this homework on the course site. Make sure that each problem (by question number) is separated from each other (i.e. you should not have the solutions to Question 1 and Question 2 on the same page).

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1. [10 pts] A group of $n \geq 4$ people stand in line to purchase movie tickets. Assume that there $r \geq 2$ men and $s \geq 2$ women, such that $r + s = n$. You should not assume that the men and women are indistinguishable from each other.

(a) How many possible lines are there such that, starting from the front of the line, there are exactly three points where the line alternates from man to woman, or woman to man?

For example, if $n = 6$, $r = 4$ and $s = 2$, one line arrangement that should be counted is (Man 1, Man 2, Woman 1, Man 3, Man 4, Woman 2). Note that there are exactly 3 points of alternation - between Man 2 and Woman 1, Woman 1 and Man 3, and Man 4 and Woman 2.

(b) Solve the above problem again with the added conditions that Man 2 must be the second man in the line, Woman 2 must be the second woman in the line, or both.

2. [10 pts] Show that the number of r-multisets of the multiset $M = \{1 \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$ is given by

$$\binom{k+r-3}{r-1} + \binom{k+r-2}{r}$$

3. [10 pts]

(a) How many strictly increasing sequences of length 10 are there whose terms are taken from 1 through 100, such that the sequences satisfy the conditions:

- the first number in the sequence is 1
- the last number in the sequence is 100
- the difference between any two numbers in the sequence is at least 2

(b) What if in addition to the conditions above, we also ask that the 3rd number in the sequence be 16, or the 8th number in the sequence be 91, or both?

4. [10 pts] Using induction, please prove the following:

(a) For all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$$

(b) For all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

5. [10 pts] We want to prove the following inequality for all $n \geq 1$.

$$\prod_{i=1}^{n} \frac{2i-1}{2i} \leq \frac{1}{\sqrt{3n}}$$

(a) Suppose that we want to prove this statement using induction, can we let our induction hypothesis simply be the above assertion? Show why this does not work.

(b) Try to instead strengthen the induction hypothesis by changing $3n$ to $3n+1$ in the above assertion. In other words, prove:

$$\prod_{i=1}^{n} \frac{2i-1}{2i} \leq \frac{1}{\sqrt{3n+1}}$$

(c) Does proving the new claim in (b) imply what you were trying to prove in part (a)?

6. [10 pts] The series

$$\sum_{k=1}^{n} \frac{1}{k}$$

is called the harmonic series. The sum of the first n numbers of the harmonic series is called the *nth harmonic number*, H_n. Thus,

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Using induction show that $H_{2^n} \geq 1 + \frac{n}{2}$. In other words, prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$$