CMSC 250 Assigned: June 6, 2017

Homework 2

Due: June 9, 2017

This assignment is due at 12PM on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified.

Each solution must be written independently by yourself - no collaboration is allowed.

In your answers, you may use results that we have proved in lecture, results from discussion sessions, and results from previous homeworks as building blocks for your solutions. We also allow you to assume the results for the parity of sums and products (i.e. you can assume odd + odd = even, even × odd = even, etc.) Everything else you will have to prove on your own!

Remember that your solutions must be typeset in LaTeX. You can find a template for this homework on the course site. Make sure that each problem (by question number) is separated from each other (i.e. you should not have the solutions to Question 1 and Question 2 on the same page).

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1. [12 pts] For each of the "proofs" below, say whether the proof is valid or invalid. If it is invalid indicate clearly as to where the logical error in the proof lies and justify why this is a logical error. If the proof is valid, you can simply say so. Just stating that the claim is false will not be awarded credit.

(a) **Claim:** All natural numbers are divisible by 3.

Proof: Suppose, for the sake of contradiction, the statement were false. Let X be the set of counterexamples, i.e.,

$X = \{x \in \mathbb{N} \mid x \text{ is not divisible by } 3\}$. The supposition that the statement is false means that $X \neq \varnothing$. Since X is a non-empty set of natural numbers, it must contain a smallest element x.

Note that $0 \notin X$ because 0 is divisible by 3. So $x \neq 0$. Now consider $x-3$. Since $x-3 < x$, $x-3 \notin X$, otherwise it would be the smallest element instead of x. Since it is not a counterexample, therefore $x-3$ is divisible by 3; that is, there is an integer a such that $x-3 = 3a$. So $x = 3a+3 = 3(a+1)$ and x is divisible by 3, contradicting $x \in X$.

(b) **Claim:** For all $n \in \mathbb{N}, 2n+1$ is a multiple of 3 $\implies$ (n^2+1) is a multiple of 3.

Proof: We will prove the contrapositive. Assume $(2n+1)$ is not a multiple of 3. There are three cases:

- If $n = 3k$, for $k \in \mathbb{N}$, then $n^2+1 = 9k^2+1$ is not a multiple of 3.

- If $n = 3k + 1$ for $k \in \mathbb{N}$, then $(2n + 1) = 6k + 3$ is a multiple of 3. We conclude that this case is not possible, since we know $(2n + 1)$ is not a multiple of 3 to begin with.
- If $n = 3k + 2$ for $k \in \mathbb{N}$, then $n^2 + 1 = 9k^2 + 12k + 5$ is not a multiple of 3.

In all cases, we have concluded $n^2 + 1$ is not a multiple of 3, so we have proved the claim.

(c) **Claim:** $\sqrt{10} + \sqrt{2} < 6$

Proof: From the claim, squaring both sides of the inequality in question gives us $12+4\sqrt{5} < 36$. If $12 + 4\sqrt{5} < 36$, then we can further simplify it into $\sqrt{5} < 6$. Squaring both sides gives us $5 < 36$, which we know is true.

(d) **Claim:** If x and y are integers then xy^2 has the same parity as x.

Proof: Assume, without loss of generality that x is even. By definition of an even integer, $x = 2k$, for some integer k. Thus

$$xy^2 = (2k)y^2 = 2(ky^2)$$

Since ky^2 is an integer, xy^2 is even and hence has the same parity as x.

2. [**10 pts**] Let t be a positive integer. Prove that

if r is irrational, then $r^{1/t}$ is irrational.

3. [**10 pts**] Consider sets A and B. Let 2^A and 2^B be the power sets of sets A and B, respectively. Prove that

$$2^{(A \cap B)} = 2^A \cap 2^B$$

4. [**10 pts**] Let $x_1, x_2, \ldots, x_{11}$ be a permutation of numbers from 1 to 11. For example, 1,3,11,10,9,5,6,4,8,2,7 would be one possible permutation.

Prove that the product

$$(x_1 - 1) \times (x_2 - 2) \times \cdots \times (x_{11} - 11)$$

is an even number.

5. [**10 pts**] We all know that $\sqrt{6}$ is a pretty crazy number. Given this, prove that $\sqrt{6}$ is irrational.

6. [**10 pts**] Let A be a set and R a relation on A that is both symmetric and antisymmetric.

(a) Prove that if a, b are distinct elements in A, then $(a, b) \notin R$.

(b) Is R reflexive? Prove your answer or give a counterexample.