CMSC 250 Assigned: July 14, 2017

Homework 10

Due: July 18, 2017

This assignment is due at 12PM on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified.

Each solution must be written independently by yourself - no collaboration is allowed.

In your answers, you may use results that we have proved in lecture, results from discussion sessions, and results from previous homeworks as building blocks for your solutions. We also allow you to assume the results for the parity of sums and products (i.e. you can assume odd + odd = even, even × odd = even, etc.) Everything else you will have to prove on your own!

Remember that your solutions must be typeset in LaTeX. You can find a template for this homework on the course site. Make sure that each problem (by question number) is separated from each other (i.e. you should not have the solutions to Question 1 and Question 2 on the same page).

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1. **[10 pts]** Let G be a graph where $\delta(G) > 2$. Prove that there exists some cycle in G, call it C, such that C consists on an even number of edges.

2. **[10 pts]** Let $a_1, a_2, \ldots, a_n$ be a list of n distinct numbers. We say that a_i and a_j are *inverted* if $i < j$ but $a_i > a_j$. The *Bubblesort* sorting algorithm swaps pairwise adjacent inverted numbers in the list until there are no more inversions, so the list is in sorted order. Suppose that the input to Bubblesort is a random permutation, equally likely to be any of the $n!$ permutations of n distinct numbers. Determine the expected number of inversions that need to be corrected by Bubblesort.

3. **[10 pts]** Sparrow and his crew are in port during a bad storm, not wanting to risk the Ruby Perl by staying at sea. In an impulse decision, a few crew members decide to go to a casino to pass the time. Sparrow was hesitant at first, wanting his crew to be off to terrorize the sea once again as soon as the storm let up. However, when he heard the news that the dreaded Red Beard* and his crew were also in town and would be at the casino, Sparrow readily joined the adventure. The two captains with their crew entered the casino at the same time and each person went to a random blackjack table (note there is no limit on how many players are at one blackjack table). Sparrow counts that there are 15 blackjack tables, and 15 members of each crew present. Hoping that the crews would mix, Sparrow asks you to compute the expected number of blackjack tables that have a member of both crews present.

* Red Beard is one of the most notorious pirate lords. Many pirates (including Sparrow) look

up to Red Beard, but many more fear him. Rumor has it his hair is red because he was once caught and ordered to be set aflame, but escaped after the flames did no harm to him. Those that are close to him simply say "he was kissed by fire".

4. [12 pts]

One night, you get an email from Ticketmaster advertising free tickets to see the Black Eyed Fleas, so you sign up and are surprised when you actually win the contest. So you go to the concert, but aren't content to just see them. Knowing that arth.u.r, a singer in the group, is a HUGE advocate for everyone to learn to code, you decide that you will find a way to talk to him about your own awesome coding experiences. After the concert, you sneak backstage and try to find your way to arth.u.r. When you finally see him, you introduce yourself as a young coder who wants to talk about Computer Science. Seeing how enthusiastic you are, he strikes up conversation and you quickly start talking about the topics in CMSC 250.

With the current topics in your mind, you ask him what his favorite type of graph is. He thinks for a minute, then responds that his favorite graphs are $(1, 3)$-trees. Unsure of what he means, you ask him to clarify what they are and why they are his favorites. He explains that a $(1, 3)$-tree is a tree all of whose vertices have degree either 1 or 3. He says that they are his favorites because they have some cool properties that are fun to prove. On a sheet of paper, he jots down his email and says if you email him with the correct proof of the following properties, he'd consider giving you and the other CMSC 250 students a free private concert. When you get home, you look at the paper and there are three things written on it:

- a link to the Gradescope submission page (which you assume is his email)
- the following two properties he wants you to prove:
  (a) If a $(1, 3)$-tree has a leaves show that it has $a - 2$ vertices of degree 3.
  (b) Let T be a $(1, 3)$-tree with at least four leaves, i.e., $a \geq 4$. Then show that there is some internal vertex that is adjacent to two leaves.

5. [10 pts] There are n people standing in line, from left to right. Suppose that all the people have different heights. A "dip" occurs at Position i in the line if the person in Position i is shorter than both the people in Position $i - 1$ and Position $i + 1$. Note that by this definition, there cannot be a "dip" at Position 1 and n.

Suppose we choose a permutation of the people in the line uniformly at random. What is the expected number of "dips" that occur in the line?

6. [10 pts] Using induction, prove that all graphs have an even number of vertices with odd degree.