CMSC 250 Assigned: June 2, 2017

Homework 1

Due: June 6, 2017

This assignment is due at 12PM on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified.

Each solution must be written independently by yourself - no collaboration is allowed.

In your answers, you may use results that we have proved in lecture, results from discussion sessions, and results from previous homeworks as building blocks for your solutions. We also allow you to assume the results for the parity of sums and products (i.e. you can assume odd + odd = even, even × odd = even, etc.) Everything else you will have to prove on your own!

Remeber that your solutions must be typeset in LaTeX. You can find a template for this homework on the course site. Make sure that each problem (by question number) is separated from each other (i.e. you should not have the solutions to Question 1 and Question 2 on the same page).

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1. [12 pts] Give answers to the following questions. There is no need to justify your answers for this question.

(a) List the members of these sets.

i. $\{x \mid x \text{ is prime and less than } 20\}$

ii. $\{x \in \mathbb{N} \mid x \leq 20 \wedge (2|x \oplus 3|x)\}$

iii. $\{x \mid (\exists y \in \mathbb{Z}^+, \exists z \in \mathbb{Z}^+, x = y(y+z)) \wedge x < 15\}$

iv. $\{x \mid x \subseteq \{a, b, c, d\} \wedge \{a, b\} \subseteq x\}$

(b) Use the set builder notation to give a description of each of these sets.

i. $\{6, 9, 12, 15, 18, 21, 24, \dots\}$

ii. $\{0, 1, 8, 27, 64, 125, \dots\}$

iii. $\{0, \frac{3}{4}, \frac{8}{9}, \frac{15}{16}, \frac{24}{25}, \dots\}$

(c) What is the cardinality of each of the following sets?

i. $\{\{a, b\}\}$

ii. $\{\{\{a\}, a\}\}$

iii. $\{a, \{a\}, \varnothing\}$

iv. $\{\{a, \{b\}\}, \{a, \{a\}\}\}$

(d) Determine whether each of the following is true or false.

i. $\{1, 2\} \subseteq \{1\}$

ii. $\varnothing \subseteq \varnothing$

iii. $\varnothing \in \varnothing$

iv. $\varnothing \subseteq \{\{\varnothing\}\}$

v. $\varnothing \subseteq 2^{\varnothing}$

vi. $\varnothing \in 2^{\varnothing}$

(e) Let A be the power set of $\{1, a, +, K\}$. What is A and $|A|$?

(f) Find two sets A and B such that $A \in B$ and $A \subseteq B$.

2. [8 pts] In logic, a tautology is a proposition that is true in every possible assignment of truth values to its propositional symbol.

For example, $(p \vee \neg p) \vee q$ is a tautology, since the expression is true regardless of the assignment of truth values to p and q.

Using a truth table, state whether the following proposition forms are a tautology . You should show your truth table, and make sure to include all intermediate logical expressions – for example, in showing the truth table for $(p \vee q) \wedge p$, you should show $p \vee q$ as a separate column on its own.

(a) $((p \rightarrow q) \vee (p \wedge \neg q)) \rightarrow (p \vee q)$

(b) $(p \wedge q) \rightarrow ((\neg p \vee p) \vee \neg q)$

3. [12 pts] Prove the following.

(a) Let $X = \{a \in \mathbb{N} \,|\, a = 3s + 2,$ for some $s \in \mathbb{N}\}$ and $Y = \{b \in \mathbb{N} \,|\, b = 6t - 1,$ for some $t \in \mathbb{N}$ and $t \geq 5\}$. Prove that $X \neq Y$.

(b) Let $X = \{a \in \mathbb{N} \mid a = 7s - 2,$ for some $s \in \mathbb{N}\}$ and $Y = \{b \in \mathbb{N} \mid b = 56t - 2,$ for some $t \in \mathbb{N}\}$. Prove that $Y \subset X$.

(c) Let $X = \{a \in \mathbb{N} \,|\, a = 4s - 2$ and $s \in \mathbb{N}$ and $s \geq 1\}$ and $Y = \{b \in \mathbb{N} \,|\, b = 4t + 2,$ for some $t \in \mathbb{N}\}$. Prove that $X = Y$.

4. [12 pts] Prove or disprove the following.

(a) For every prime p, either $p + 2$ is a prime, or $p + 3$ is a prime.

(b) Given any $x, y, z \in \mathbb{R}$, if $x - y$ is odd and $y - z$ is even, then $x - z$ is odd.

(c) $\forall x \in \mathbb{N}$, $x^3 - 2x^2 + 3x + 3$ is odd.

(d) $\forall a, b, x, y, d \in \mathbb{Z}, (d \mid a \wedge d \mid b) \rightarrow d \mid (ax + by)$.

5. [12 pts] Let A, B and C be sets. Prove or disprove the following.

(a) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$

(b) $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$.

(c) $A \setminus C = (A \setminus (B \cup C)) \cup ((A \cap B) \setminus C)$

(d) A and B are disjoint $\implies$ $A \times B$ and $B \times A$ are disjoint.

6. [10 pts] Suppose that $a, b \in (\mathbb{R} \setminus \{0\})$. Prove that if $a < \frac{1}{a} < b < \frac{1}{b}$, then $a < -1$.