CMSC 250: Discrete Structures

Summer 2017

Discussion Session 6

July 12, 2017

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1. Answer the following problems about graphs.

   (a) How many edges are there in a complete graph (every possible edge exists) with n vertices?

   (b) How many edges are there in a forest with n vertices and k connected components?

2. Prove that, in a graph, there must exist a path between any vertex with odd degree and some other vertex with odd degree.

3. Prove that G or the complement of G is connected.

   Note: For any graph $G = (V, E)$, the complement graph of G, denoted $\overline{G} = (V, \overline{E})$, is defined as follows:

   - G and $\overline{G}$ share the same vertex set V
   - $\overline{E}$ contains all of the possible edges that are not in E, and none of the edges that are in E. Formally, $\overline{E} = \{S \in 2^V \mid |S| = 2\} \setminus E$.

4. Show why the following proof is incorrect.

   Claim: In a graph G with at least 2 vertices, if $\delta(G) \geq 1$ then G is connected

   We induct on the number of vertices, n.

   Base Case: $n = 2$. Since $\delta(G) \geq 1$, this graph is the graph of 2 vertices connected by an edge. It is connected.

   Induction Hypothesis: Given a graph G with k vertices ($k \in \mathbb{Z}$, $k \geq 2$), if $\delta(G) \geq 1$, then G is connected.

   Induction Step: Consider a graph G with k vertices such that $\delta(G) \geq 1$. Then by the induction hypothesis, G is connected. Now add a vertex u to form G'. In order for u to have degree at least 1, it needs to be connected to some other vertex v in G. G is connected, there exists a path from v to every vertex in G, and since we now have the edge $\{u, v\}$, there now exists a path from u to every vertex in G. Thus, the graph G' with $k + 1$ vertices is connected.

5. Let's say we are playing a game. I first roll a fair 6-sided die. If the number that shows is divisible by 3, I roll again and I pay you the dollar amount that shows up on the second roll. If not, then I flip a fair coin. If it's tails, I take 10 dollars from you, and if it's heads, I pay you 5 dollars. What is your expected payoff?

6. Dhruv, as a busy college student, hasn't done laundry in weeks. In particular, he realizes that he has no more socks to wear, so he goes to the laundry room and throws in his 2 distinct pairs of socks. However, the machine is broken, and so it only returns 2 of his socks at random!

   (a) What is the expected number of pairs that he can wear now?

   (b) What if he throws in 8 pairs and only gets 12 socks back?

   (c) What if he throws in n pairs and only gets k ($k > 1$) sock back?