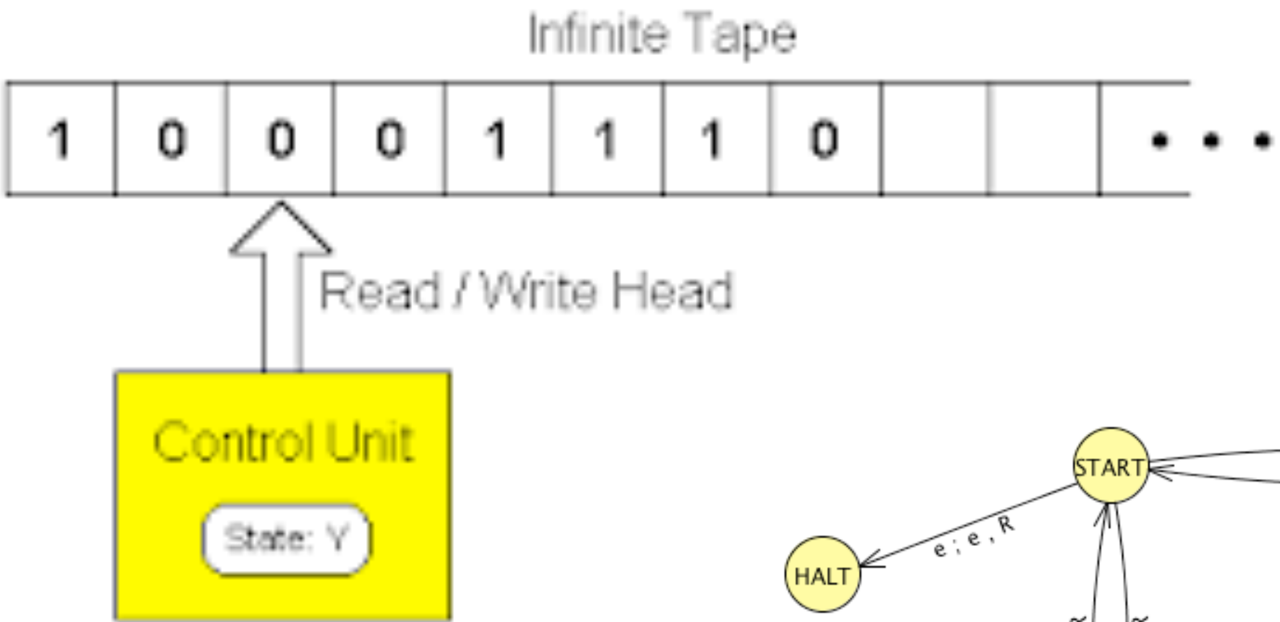
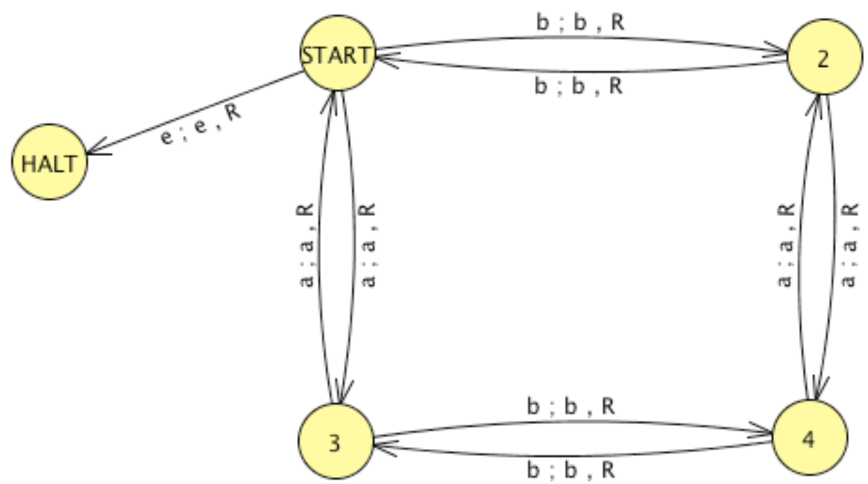
# CMSC 330: Organization of Programming Languages

## Lambda Calculus

# Turing Machine



# Turing Completeness

- Turing machines are the most powerful description of computation possible
  - They define the Turing-computable functions
- A programming language is Turing complete if
  - It can map every Turing machine to a program
  - A program can be written to emulate a Turing machine
  - It is a superset of a known Turing-complete language
- Most powerful programming language possible
  - Since Turing machine is most powerful automaton

# Programming Language Expressiveness

- So what language features are needed to express all computable functions?
  - What's a minimal language that is Turing Complete?
- Observe: some features exist just for convenience
  - Multi-argument functions        foo ( a, b, c )
    - Use currying or tuples
  - Loops        while (a < b) …
    - Use recursion
  - Side effects        a := 1
    - Use functional programming pass "heap" as an argument to each function, return it when with function's result:

      effectful : 'a → 's → ('s * 'a)

# Programming Language Expressiveness

- It is not difficult to achieve Turing Completeness
  - Lots of things are ‘accidentally’ TC
- Some fun examples:
  - x86_64 `mov` instruction
  - Minecraft
  - Magic: The Gathering
  - Java Generics
- There’s a whole cottage industry of proving things to be TC
- But: What is a “core” language that is TC?

# Lambda Calculus (λ-calculus)

- Proposed in 1930s by
  - Alonzo Church
    (born in Washington DC!)
- Formal system
  - Designed to investigate functions & recursion
  - For exploration of foundations of mathematics
- Now used as
  - Tool for investigating computability
  - Basis of functional programming languages
    - Lisp, Scheme, ML, OCaml, Haskell…

# Why Study Lambda Calculus?

- It is a “core” language
  - Very small but still Turing complete
- But with it can explore general ideas
  - Language features, semantics, proof systems, algorithms, …
- Plus, higher-order, anonymous functions (aka *lambdas*) are now very popular!
  - C++ (C++11), PHP (PHP 5.3.0), C# (C# v2.0), Delphi (since 2009), Objective C, Java 8, Swift, Python, Ruby (Procs), … (and functional languages like OCaml, Haskell, F#, …)
  - Excel, as of 2021!

# Lambda Calculus Syntax

- A lambda calculus expression is defined as

| | |
|---|---|
| e ::= x | **variable** |
| \| λx.e | **abstraction** (fun def) |
| \| e e | **application** (fun call) |

  - This grammar describes ASTs; not for parsing - ambiguous!
  - Lambda expressions also known as lambda **terms**

- λx.e is like `(fun x -> e)` in OCaml

That's it! Nothing but higher-order functions

# Three Conventions

- Scope of λ extends as far right as possible
  - Subject to scope delimited by parentheses
  - λx. λy.x y is same as λx.(λy.(x y))
- Function application is left-associative
  - x y z is (x y) z
  - Same rule as OCaml
- As a convenience, we use the following “syntactic sugar” for local declarations
  - let x = e1 in e2 is short for (λx.e2) e1

# Quiz #1

**`λx.(y z)`** and **`λx.y z`** are equivalent

A. True
B. False

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# Quiz #2

This term is equivalent to which of the following?

```
λx.x a b
```

A. `(λx.x) (a b)`
B. `(((λx.x) a) b)`
C. `λx.(x (a b))`
D. `(λx.((x a) b))`

# Quiz #2

This term is equivalent to which of the following?

**`λx.x a b`**

A. `(λx.x) (a b)`
B. `(((λx.x) a) b)`
C. `λx.(x (a b))`
D. `(λx.((x a) b))`

# Lambda Calculus Semantics

- Evaluation: All that's involved are function calls
  (λx.e1) e2
  - Evaluate e1 with x replaced by e2
- This application is called beta-reduction
  - (λx.e1) e2 → e1[x:=e2]
    - e1[x:=e2] is e1 with occurrences of x replaced by e2
    - This operation is called *substitution*
      - **Replace** formals with actuals
      - Instead of using environment to map formals to actuals
  - We allow reductions to occur *anywhere* in a term
    - Order reductions are applied does not affect final value!
- When a term cannot be reduced further it is in beta normal form

# Beta Reduction Example

- (λx.λz.x z) y

  → (λx.(λz.(x z))) y  // since λ extends to right

  → (λx.(λz.(x z))) y  // apply (λx.e1) e2 → e1[x:=e2]

  // where e1 = λz.(x z), e2 = y

  → λz.(y z)  // final result

Parameters
- Formal
- Actual

- Equivalent OCaml code
  - (fun x -> (fun z -> (x z))) y → fun z -> (y z)

# Beta Reductions (CBV)

- (λx.x) z → z
- (λx.y) z → y
- (λx.x y) z → z y
  - A function that applies its argument to y

# Beta Reductions (CBV)

- $(\lambda x.x\ y)\ (\lambda z.z) \rightarrow\ (\lambda z.z)\ y \rightarrow y$

- $(\lambda x.\lambda y.x\ y)\ z \rightarrow\ \lambda y.z\ y$
  - A curried function of two arguments
  - Applies its first argument to its second

- $(\lambda x.\lambda y.x\ y)\ (\lambda z.zz)\ x \rightarrow (\lambda y.(\lambda z.zz)y)x \rightarrow (\lambda z.zz)x \rightarrow x\ x$