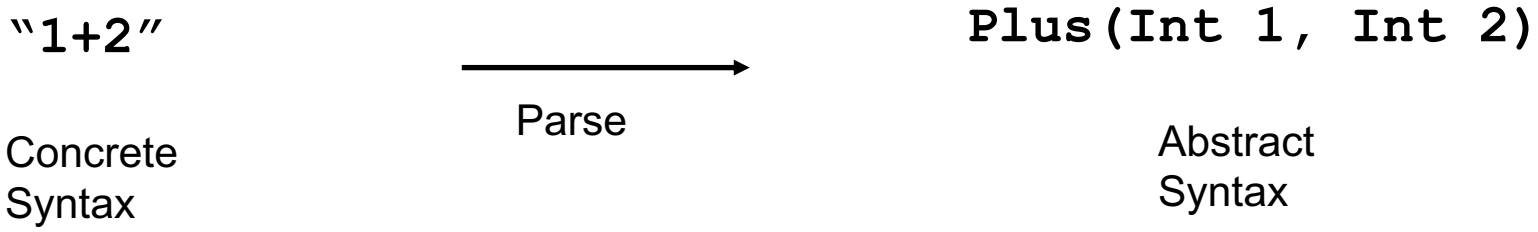
CMSC 330: Organization of Programming Languages

Operational Semantics

Formal Semantics of a Prog. Lang.

- Mathematical description of the meaning of programs written in that language
  - What a program computes, and what it does



- What does `Plus(Int 1, Int 2)` mean?

Operational semantics

- Define how programs execute
  - Often on an abstract machine (mathematical model of computer)
  - Analogous to interpretation
- We will define an operational semantics for Micro-Ocaml
  - And develop an interpreter for it, along the way
- Approach: use rules to define a judgment

$$e \Rightarrow v$$

Micro-OCaml Expression Grammar

e ::= *x* | *n* | *e* + *e* | let *x* = *e1* in *e2*

Corresponding AST:

```
type id = string
type exp =
    | Ident of id              (* x *)
    | Num of int               (* n *)
    | Plus of exp * exp        (* e+e *)
    | Let of id * exp * exp    (* let x=e1 in e2 *)
```

Defining the Semantics

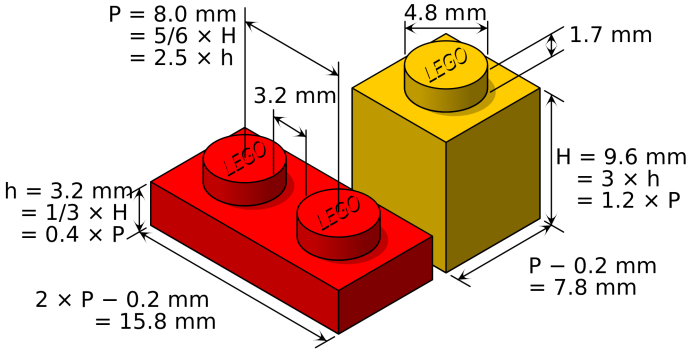
- Use rules to define judgment **`e`** ⇒ **`v`**
- Inference Rules

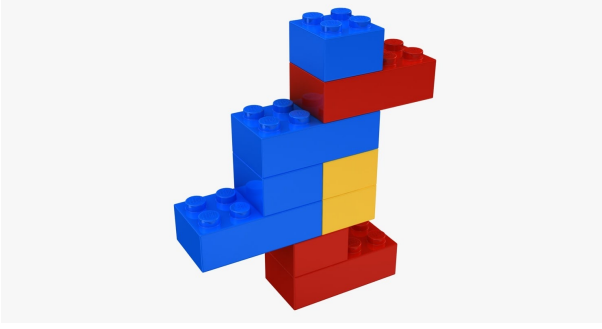
$$\frac{H_1,\ H_2\ \ldots\ H_n}{C}$$

$$\frac{\forall x\left(Man(x) \rightarrow Mortal(x)\right) \qquad Man(Socrates)}{\therefore \quad Mortal(Socrates)}$$

H_1 ^ H_2 ^ … H_n ⇒ C

Rules are Lego Blocks





Rules of Inference: Num and Sum

$$\frac{}{n \Rightarrow n}$$

$$\frac{e1 \Rightarrow n1 \qquad e2 \Rightarrow n2 \qquad n3 \text{ is } n1+n2}{e1 + e2 \Rightarrow n3}$$

```
match e with
| Num n -> n
| Plus (e1,e2) ->
      let n1 = eval e1 in
      let n2 = eval e2 in
      let n3 = n1 + n2 in
      n3
```

Rules of Inference: Let

$$\frac{e1 \Rightarrow v1 \quad e2\{v1/x\} \Rightarrow v2}{\texttt{let } x \texttt{ = } e1 \texttt{ in } e2 \Rightarrow v2}$$

```
match e with
| Let (x,e1,e2) ->
      let v1 = eval e1 in
      let e2' = subst v1 x e2 in
      let v2 = eval e2'
  in v2
```

Derivations

- When we apply rules to an expression in succession, we produce a derivation
  - It's a kind of tree, rooted at the conclusion

- Produce a derivation by goal-directed search
  - Pick a rule that could prove the goal
  - Then repeatedly apply rules on the corresponding hypotheses
    - Goal: Show that `let x = 4 in x+3` ⇒ `7`

Derivations

$$\frac{}{n \Rightarrow n} \qquad \frac{e1 \Rightarrow n1 \quad e2 \Rightarrow n2 \quad n3 \text{ is } n1+n2}{e1 + e2 \Rightarrow n3}$$

$$\frac{e1 \Rightarrow v1 \quad e2\{v1/x\} \Rightarrow v2}{\texttt{let } x \texttt{ = } e1 \texttt{ in } e2 \Rightarrow v2}$$

Goal: show that
`let x = 4 in x+3` ⇒ `7`

$$\frac{\dfrac{}{4 \Rightarrow 4} \quad \dfrac{}{3 \Rightarrow 3} \quad 7 \text{ is } 4+3}{\dfrac{4 \Rightarrow 4 \qquad 4+3 \Rightarrow 7}{\texttt{let x = 4 in x+3} \Rightarrow 7}}$$

Quiz 1

What is derivation of the following judgment?

$$2 + (3 + 8) \Rightarrow 13$$

(a)

$$\frac{2 \Rightarrow 2 \qquad 3 + 8 \Rightarrow 11}{2 + (3 + 8) \Rightarrow 13}$$

(b)

$$\frac{\dfrac{8 \Rightarrow 8 \qquad 3 \Rightarrow 3 \qquad 11 \text{ is } 3+8}{3 + 8 \Rightarrow 11} \qquad 2 \Rightarrow 2 \qquad 13 \text{ is } 2+11}{2 + (3 + 8) \Rightarrow 13}$$

(c)

$$\frac{\dfrac{3 \Rightarrow 3 \qquad 8 \Rightarrow 8}{3 + 8 \Rightarrow 11} \qquad 2 \Rightarrow 2}{2 + (3 + 8) \Rightarrow 13}$$

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(c)

$$\frac{\dfrac{3 \Rightarrow 3 \qquad 8 \Rightarrow 8}{3 + 8 \Rightarrow 11} \qquad 2 \Rightarrow 2}{2 + (3 + 8) \Rightarrow 13}$$

Definitional Interpreter

- The style of rules lends itself directly to the implementation of an interpreter as a recursive function

```
let rec eval (e:exp):value =
  match e with
    Ident x -> (* no rule *)
     failwith "no value"
  | Num n -> n
  | Plus (e1,e2) ->
     let n1 = eval e1 in
     let n2 = eval e2 in
     let n3 = n1+n2 in
     n3
  | Let (x,e1,e2) ->
     let v1 = eval e1 in
     let e2' = subst v1 x e2 in
     let v2 = eval e2' in v2
```

$$\frac{}{n \Rightarrow n}$$

$$\frac{e1 \Rightarrow n1 \quad e2 \Rightarrow n2 \quad n3 \text{ is } n1+n2}{e1 + e2 \Rightarrow n3}$$

$$\frac{e1 \Rightarrow v1 \quad e2\{v1/x\} \Rightarrow v2}{\texttt{let } x \texttt{ = } e1 \texttt{ in } e2 \Rightarrow v2}$$

Derivations = Interpreter Call Trees

$$\dfrac{\dfrac{}{\texttt{4} \Rightarrow \texttt{4}} \quad \dfrac{}{\texttt{3} \Rightarrow \texttt{3}} \quad \dfrac{}{\texttt{7 is 4+3}}}{\dfrac{\dfrac{}{\texttt{4} \Rightarrow \texttt{4}} \quad \dfrac{}{\texttt{4+3} \Rightarrow \texttt{7}}}{\texttt{let x = 4 in x+3} \Rightarrow \texttt{7}}}$$

Has the same shape as the recursive call tree of the interpreter:

$$\dfrac{\dfrac{}{\texttt{eval Num 4} \Rightarrow 4} \quad \dfrac{}{\texttt{eval Num 3} \Rightarrow 3} \quad \dfrac{}{\texttt{7 is 4+3}}}{\dfrac{\dfrac{}{\texttt{eval Num 4} \Rightarrow 4} \quad \dfrac{}{\texttt{eval (subst 4 "x" Plus(Ident("x"),Num 3))} \Rightarrow 7}}{\texttt{eval Let("x",Num 4,Plus(Ident("x"),Num 3))} \Rightarrow 7}}$$

Semantics Defines Program Meaning

- **e** ⇒ **v** holds if and only if a *proof* can be built
  - Proofs are derivations: axioms at the top, then rules whose hypotheses have been proved to the bottom
  - No proof means there *exists no* **v** *for which* **e** ⇒ **v**
- Proofs can be constructed bottom-up
  - In a goal-directed fashion
- Thus, function eval **e** = {**v** | **e** ⇒ **v** }
  - Determinism of semantics implies at most one element for any **e**
- So: Expression **e** *means* **v**

Environment-style Semantics

- So far, semantics used substitution to handle variables
  - As we evaluate, we replace all occurrences of a variable ***x*** with values it is bound to

- An alternative semantics, closer to a real implementation, is to use an environment
  - As we evaluate, we maintain an explicit map from variables to values, and look up variables as we see them

Environments

- Mathematically, an environment is a partial function from identifiers to values
  - If A is an environment, and ***x*** is an identifier, then A(***x***) can either be
    - a value ***v*** (intuition: the value of the variable stored on the stack)
    - undefined (intuition: the variable has not been declared)
- An environment can visualized as a table
  - If A is

Id	Val
`x`	0
`y`	2

  - then A(`x`) is `0`, A(`y`) is `2`, and A(`z`) is undefined

Notation, Operations on Environments

- • is the empty environment
- A,***x***:***v*** is the environment that extends A with a mapping from ***x*** to ***v***
  - Sometimes just write ***x***:***v*** instead of •,***x***:***v*** for brevity
- Lookup A(***x***) is defined as follows

•(***x***) = undefined

$$(A, \mathbf{y}{:}v)(x) = \begin{cases} v & \text{if } x = \mathbf{y} \\ A(x) & \text{if } x <> \mathbf{y} \text{ and } A(x) \text{ defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Definitional Interpreter: Environments

```
type env = (id * value) list

let extend env x v = (x,v)::env

let rec lookup env x =
  match env with
     [] -> failwith "undefined"
  | (y,v)::env' ->
     if x = y then v
     else lookup env' x
```

An environment is just a list of mappings, which are just pairs of variable to value - called an association list

Semantics with Environments

- The environment semantics changes the judgment

  $$e \Rightarrow v$$

  to be

  $$A;\ e \Rightarrow v$$

  where A is an environment
  - Idea: A is used to give values to the identifiers in `e`

Environment-style Rules

$$\frac{A(x) = v}{A;\ x \Rightarrow v}$$

Look up variable x in environment A

$$\frac{}{A;\ n \Rightarrow n}$$

$$\frac{A;\ e1 \Rightarrow v1 \qquad A, x : v1;\ e2 \Rightarrow v2}{A;\ \texttt{let}\ x\ \texttt{=}\ e1\ \texttt{in}\ e2 \Rightarrow v2}$$

Extend environment A with mapping from x to $v1$

$$\frac{A;\ e1 \Rightarrow n1 \qquad A;\ e2 \Rightarrow n2 \qquad n3 \text{ is } n1+n2}{A;\ e1 + e2 \Rightarrow n3}$$

Definitional Interpreter: Evaluation

```
let rec eval env e =
  match e with
    Ident x -> lookup env x
  | Num n -> n
  | Plus (e1,e2) ->
     let n1 = eval env e1 in
     let n2 = eval env e2 in
     let n3 = n1+n2 in
     n3
  | Let (x,e1,e2) ->
     let v1 = eval env e1 in
     let env' = extend env x v1 in
     let v2 = eval env' e2 in v2
```

Quiz 2

What is a derivation of the following judgment?

$$\bullet ; \texttt{let x=3 in x+2} \Rightarrow 5$$

(a)

$$\frac{x \Rightarrow 3 \quad 2 \Rightarrow 2 \quad 5 \text{ is } 3+2}{\dfrac{3 \Rightarrow 3 \quad x+2 \Rightarrow 5}{\texttt{let x=3 in x+2} \Rightarrow 5}}$$

(b)

$$\frac{\dfrac{x:3; x \Rightarrow 3 \quad x:3; 2 \Rightarrow 2 \quad 5 \text{ is } 3+2}{\bullet;3 \Rightarrow 3 \quad x:3; x+2 \Rightarrow 5}}{\bullet; \texttt{let x=3 in x+2} \Rightarrow 5}$$

(c)

$$\frac{x:2; x \Rightarrow 3 \quad x:2; 2 \Rightarrow 2 \quad 5 \text{ is } 3+2}{\bullet; \texttt{let x=3 in x+2} \Rightarrow 5}$$

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(c)

$$\frac{x:2 ; x \Rightarrow 3 \quad x:2 ; 2 \Rightarrow 2 \quad 5 \text{ is } 3+2 \quad \text{----}}{\bullet ; \texttt{let x=3 in x+2} \Rightarrow 5}$$

(b)

$$\frac{x:3 ; x \Rightarrow 3 \quad x:3 ; 2 \Rightarrow 2 \quad 5 \text{ is } 3+2}{\bullet ; 3 \Rightarrow 3 \quad x:3 ; x+2 \Rightarrow 5} \qquad \frac{}{\bullet ; \texttt{let x=3 in x+2} \Rightarrow 5}$$

Adding Conditionals to Micro-OCaml

```
e ::= x | v | e + e | let x = e in e
    | eq0 e | if e then e else e

v ::= n | true | false
```

- In terms of interpreter definitions:

```
type exp =
  | Val of value
  | … (* as before *)
  | Eq0 of exp
  | If of exp * exp * exp
```

```
type value =
    Int of int
  | Bool of bool
```

Rules for Eq0 and Booleans

$$\frac{}{A;\ \texttt{true} \Rightarrow \texttt{true}}$$

$$\frac{A;\ e \Rightarrow 0}{A;\ \texttt{eq0}\ e \Rightarrow \texttt{true}}$$

$$\frac{}{A;\ \texttt{false} \Rightarrow \texttt{false}}$$

$$\frac{A;\ e \Rightarrow v \quad v \neq 0}{A;\ \texttt{eq0}\ e \Rightarrow \texttt{false}}$$

Rules for Conditionals

$$\frac{A;\ e1 \Rightarrow \texttt{true} \qquad A;\ e2 \Rightarrow v}{A;\ \texttt{if}\ e1\ \texttt{then}\ e2\ \texttt{else}\ e3 \Rightarrow v}$$

$$\frac{A;\ e1 \Rightarrow \texttt{false} \qquad A;\ e3 \Rightarrow v}{A;\ \texttt{if}\ e1\ \texttt{then}\ e2\ \texttt{else}\ e3 \Rightarrow v}$$

- Notice that only one branch is evaluated

Quiz 3

What is the derivation of the following judgment?

$$\bullet;\ \texttt{if eq0 3-2 then 5 else 10} \Rightarrow 10$$

(a)

$$\frac{\dfrac{}{\bullet;\ 3 \Rightarrow 3} \quad \dfrac{}{\bullet;\ 2 \Rightarrow 2} \quad 3\text{-}2 \text{ is } 1}{\bullet;\ \texttt{eq0 3-2} \Rightarrow \texttt{false}} \qquad \frac{}{\bullet;\ 10 \Rightarrow 10}$$
$$\bullet;\ \texttt{if eq0 3-2 then 5 else 10} \Rightarrow 10$$

(b)

$$\frac{\dfrac{}{3 \Rightarrow 3} \quad \dfrac{}{2 \Rightarrow 2} \quad 3\text{-}2 \text{ is } 1}{\texttt{eq0 3-2} \Rightarrow \texttt{false}} \qquad \frac{}{10 \Rightarrow 10}$$
$$\texttt{if eq0 3-2 then 5 else 10} \Rightarrow 10$$

(c)

$$\frac{\dfrac{\dfrac{}{\bullet;\ 3 \Rightarrow 3} \quad \dfrac{}{\bullet;\ 2 \Rightarrow 2} \quad 3\text{-}2 \text{ is } 1}{\bullet;\ 3\text{-}2 \Rightarrow 1} \quad 1 \neq 0}{\bullet;\ \texttt{eq0 3-2} \Rightarrow \texttt{false}} \qquad \frac{}{\bullet;\ 10 \Rightarrow 10}$$
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Quiz 3

What is the derivation of the following judgment?

$$\bullet;\ \texttt{if eq0 3-2 then 5 else 10} \Rightarrow 10$$

(a)

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(b)

$$\frac{\dfrac{3 \Rightarrow 3 \quad 2 \Rightarrow 2 \quad 3\text{-}2 \text{ is } 1}{\texttt{eq0 3-2} \Rightarrow \texttt{false}} \qquad 10 \Rightarrow 10}{\texttt{if eq0 3-2 then 5 else 10} \Rightarrow 10}$$

(c)

$$\frac{\dfrac{\dfrac{\bullet;\ 3 \Rightarrow 3 \quad \bullet;\ 2 \Rightarrow 2 \quad 3\text{-}2 \text{ is } 1}{\bullet;\ 3\text{-}2 \Rightarrow 1} \quad 1 \neq 0}{\bullet;\ \texttt{eq0 3-2} \Rightarrow \texttt{false}} \qquad \bullet;\ 10 \Rightarrow 10}{\bullet;\ \texttt{if eq0 3-2 then 5 else 10} \Rightarrow 10}$$

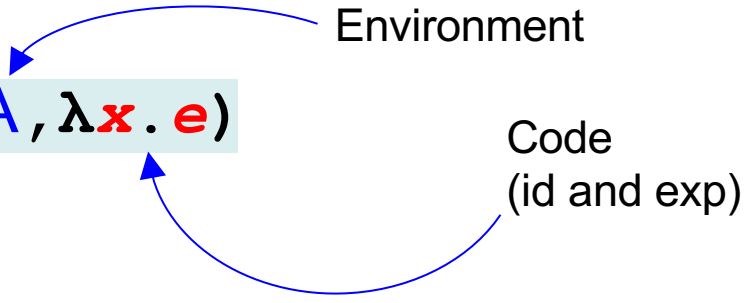
Updating the Interpreter

```
let rec eval env e =
  match e with
    Ident x -> lookup env x
  | Val v -> v
  | Plus (e1,e2) ->
     let Int n1 = eval env e1 in
     let Int n2 = eval env e2 in
     let n3 = n1+n2 in
     Int n3
  | Let (x,e1,e2) ->
     let v1 = eval env e1 in
     let env' = extend env x v1 in
     let v2 = eval env' e2 in v2
  | Eq0 e1 ->
     let Int n = eval env e1 in
     if n=0 then Bool true else Bool false
  | If (e1,e2,e3) ->
     let Bool b = eval env e1 in
     if b then eval env e2
     else eval env e3
```

Adding Closures to Micro-OCaml

```
e ::= x | v | e + e | let x = e in e
    | eq0 e | if e then e else e
    | e e | fun x -> e

v ::= n | true | false | (A, λx.e)
```



- In terms of interpreter definitions:

```
type exp =
  | Val of value
  | If of exp * exp * exp
    … (* as before *)
  | Call of exp * exp
  | Fun of id * exp
```

```
type value =
    Int of int
  | Bool of bool
  | Closure of env * id * exp
```

Rule for Closures: Lexical/Static Scoping

$$\frac{}{A;\ \texttt{fun}\ x\ \texttt{->}\ e\ \Rightarrow\ (A, \lambda x.e)}$$

$$\frac{A;\ e1 \Rightarrow (A', \lambda x.e) \qquad A;\ e2 \Rightarrow v1 \qquad A', x{:}v1;\ e \Rightarrow v}{A;\ e1\ \ e2\ \Rightarrow\ v}$$

- Notice
  - Creating a closure captures the current environment A
  - A call to a function
    - evaluates the body of the closure's code ***e*** with function closure's environment A' extended with parameter ***x*** bound to argument ***v1***

Rule for Closures: Dynamic Scoping

$$\frac{}{A;\ \texttt{fun}\ x\ \texttt{->}\ e\ \Rightarrow\ (\bullet, \lambda x.e)}$$

$$\frac{A;\ e1 \Rightarrow (\bullet, \lambda x.e) \qquad A;\ e2 \Rightarrow v1 \qquad A, x{:}v1;\ e \Rightarrow v}{A;\ e1\ \ e2 \Rightarrow v}$$

- Notice
  - Creating a closure ignores the current environment A
  - A call to a function
    - evaluates the body of the closure's code ***e*** with the current environment A extended with parameter ***x*** bound to argument ***v1***

Scaling up

- Operational semantics can handle full languages
  - With records, recursive variant types, objects, first-class functions, and more
- Provides a concise notation for explaining what a language does. Clearly shows:
  - Evaluation order
  - Call-by-value vs. call-by-name
  - Static scoping vs. dynamic scoping
  - ... We may look at more of these later

Scaling up: Lego City