PCA

CMSC 422

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Unsupervised Learning

- Discovering hidden structure in data
- What algorithms do we know for unsupervised learning?
  - K-Means Clustering
- Today: how can we learn better representations of our data points?

Dimensionality Reduction

- Goal: extract hidden lower-dimensional structure from high dimensional datasets

- Why?
  - To visualize data more easily
  - To remove noise in data
  - To lower resource requirements for storing/processing data
  - To improve classification/clustering

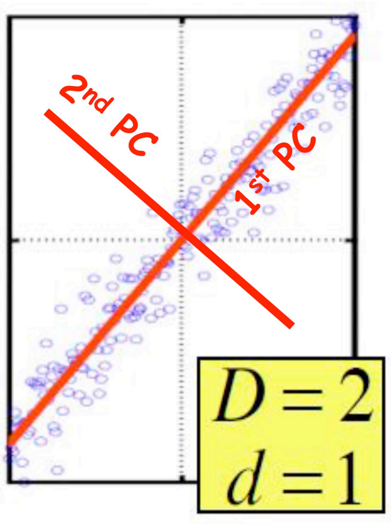
- Linear algebra review:
  - Matrix decomposition with eigenvectors and eigenvalues

Principal Component Analysis

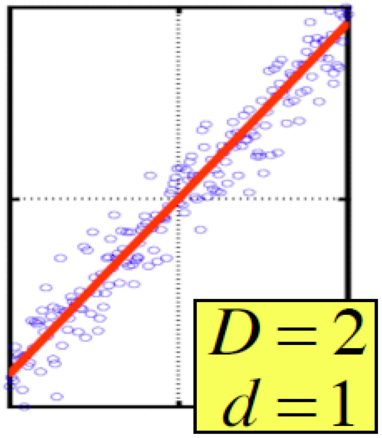
- Goal: Find a **projection** of the data onto directions that **maximize variance** of the original data set
  - Intuition: those are directions in which most information is encoded

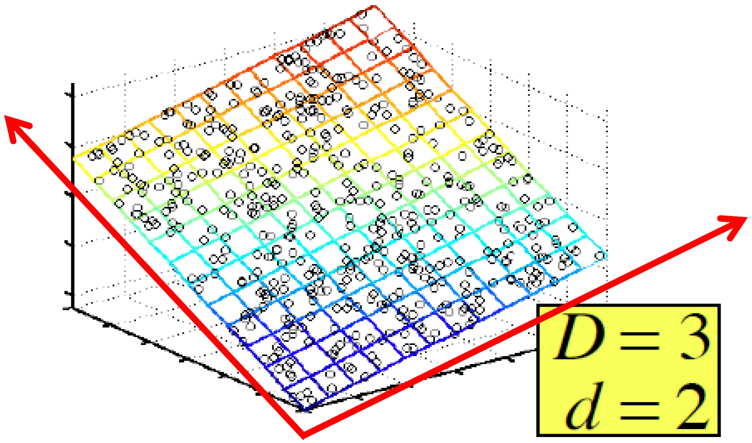
- Definition: **Principal Components** are orthogonal directions that capture most of the variance in the data

PCA: finding principal components



- 1st PC
  - Projection of data points along 1st PC discriminates data most along any one direction
- 2nd PC
  - next orthogonal direction of greatest variability
- And so on…





Examples of data points in D dimensional space that can be effectively represented in a d-dimensional subspace (d < D)

PCA: notation

- Data points
  - Represented by matrix X of size NxD
  - Let's assume data is centered

- Principal components are d vectors: $v_1, v_2, \ldots v_d$

  $v_i . v_j = 0, i \neq j$ and $v_i . v_i = 1$

- The sample variance data projected on vector v is $\sum_{i=1}^{n} ({x_i}^T v)^2 = (Xv)^T (Xv)$

PCA formally

- Finding vector that maximizes sample variance of projected data:

  $$argmax_v \, v^T X^T X v \text{ such that } v^T v = 1$$

- A constrained optimization problem
  - Lagrangian folds constraint into objective: $argmax_v \, v^T X^T X v - \lambda(v^T v - 1)$
  - Solutions are vectors v such that $X^T X v = \lambda v$
    - i.e. eigenvectors of $X^T X$ (sample covariance matrix)

PCA formally

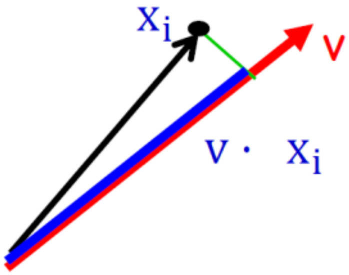
- The eigenvalue λ denotes the amount of variability captured along dimension v
  - Sample variance of projection $v^T X^T X v = \lambda$

- If we rank eigenvalues from large to small
  - The 1st PC is the eigenvector of $X^T X$ associated with largest eigenvalue
  - The 2nd PC is the eigenvector of $X^T X$ associated with 2nd largest eigenvalue
  - …

Alternative interpretation of PCA

- PCA finds vectors v such that projection on to these vectors minimizes reconstruction error

$$\frac{1}{n}\sum_{i=1}^{n}\|\mathbf{x}_i - (\mathbf{v}^T\mathbf{x}_i)\mathbf{v}\|^2$$



Resulting PCA algorithm

Algorithm 36 PCA($\mathbf{D}$, K)

1: $\boldsymbol{\mu} \leftarrow \text{MEAN}(\mathbf{X})$ // compute data mean for centering

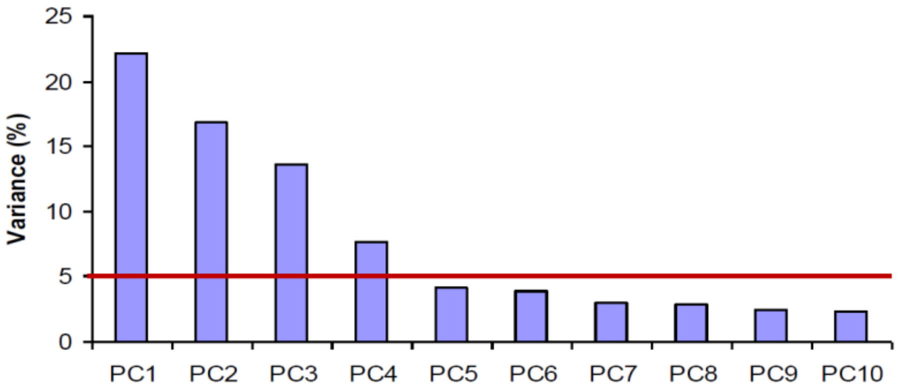
2: $\mathbf{D} \leftarrow \left(\mathbf{X} - \boldsymbol{\mu}\mathbf{1}^\top\right)^\top \left(\mathbf{X} - \boldsymbol{\mu}\mathbf{1}^\top\right)$ // compute covariance, $\mathbf{1}$ is a vector of ones

3: $\{\lambda_k, \boldsymbol{u}_k\} \leftarrow$ top K eigenvalues/eigenvectors of $\mathbf{D}$

4: **return** $(\mathbf{X} - \boldsymbol{\mu}\mathbf{1})\,\mathbf{U}$ // project data using $\mathbf{U}$

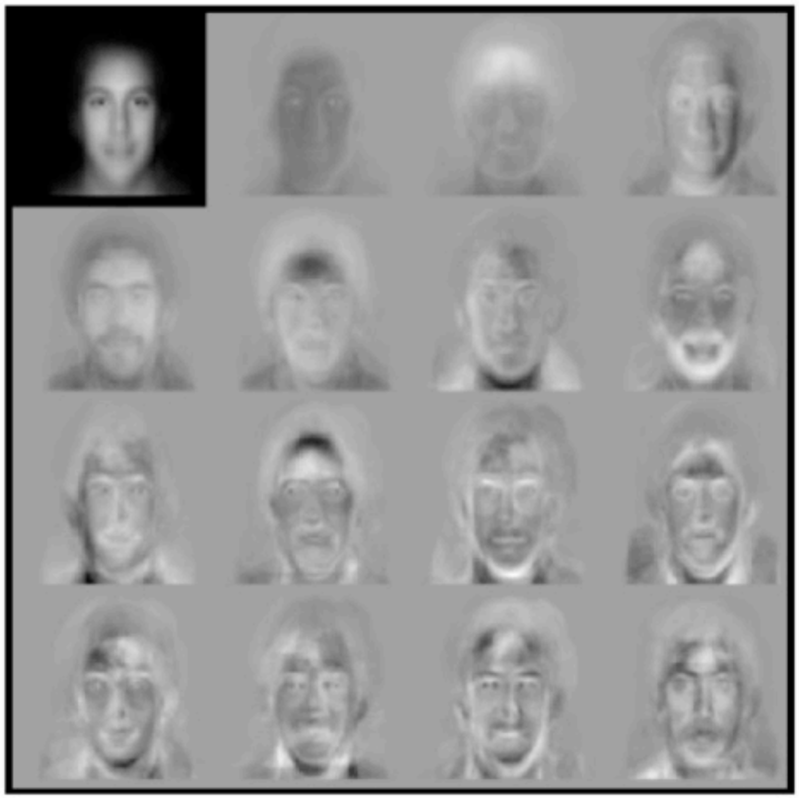
How to choose the hyperparameter K?

- i.e. the number of dimensions



- We can ignore the components of smaller significance

An example: Eigenfaces



Eigenfaces from 7562 images:

top left image is linear combination of rest.

Sirovich & Kirby (1987)
Turk & Pentland (1991)

PCA pros and cons

- Pros
  - Eigenvector method
  - No tuning of the parameters
  - No local optima

- Cons
  - Only based on covariance (2^{nd} order statistics)
  - Limited to linear projections

What you should know

- Principal Components Analysis
  - Goal: Find a **projection** of the data onto directions that **maximize variance** of the original data set
  - PCA **optimization objectives** and resulting **algorithm**
  - Why this is useful!