

Support Vector Machines

CMSC 422

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Slides adapted from MARINE CARPUAT

Today's topics

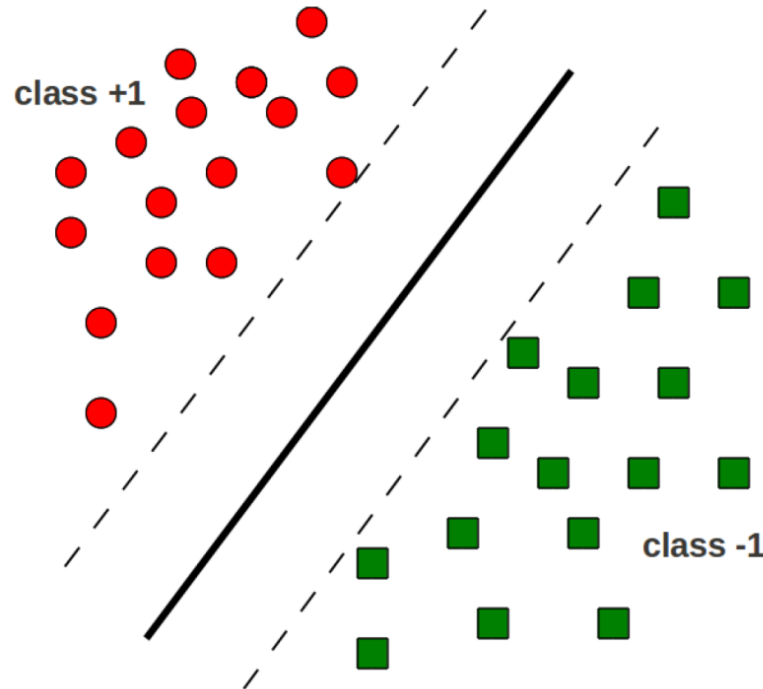
- SVMs
- HW7 posted

Back to linear classification

- So far: we've seen that kernels can help capture non-linear patterns in data while keeping the advantages of a linear classifier
- Support Vector Machines
 - A hyperplane-based classification algorithm
 - Highly influential
 - Backed by solid theoretical grounding (Vapnik & Cortes, 1995)
 - Easy to kernelize

The Maximum Margin Principle

- Find the hyperplane with **maximum separation margin** on the training data



Margin of a data set \mathbf{D}

$$\text{margin}(\mathbf{D}, \mathbf{w}, b) = \begin{cases} \min_{(x,y) \in \mathbf{D}} y(\mathbf{w} \cdot \mathbf{x} + b) & \text{if } \mathbf{w} \text{ separates } \mathbf{D} \\ -\infty & \text{otherwise} \end{cases} \quad (3.8)$$

Distance between the hyperplane (\mathbf{w}, b) and the nearest point in \mathbf{D}

$$\text{margin}(\mathbf{D}) = \sup_{\mathbf{w}, b} \text{margin}(\mathbf{D}, \mathbf{w}, b) \quad (3.9)$$

Largest attainable margin on \mathbf{D}

Support Vector Machine (SVM)

A hyperplane based linear classifier defined by \mathbf{w} and b

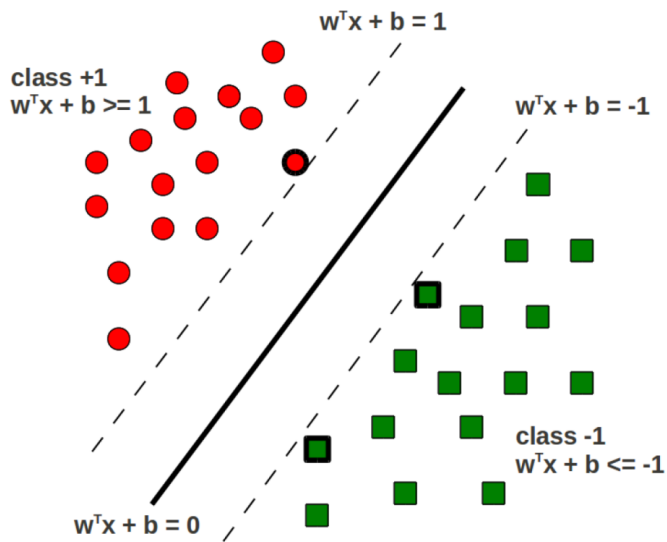
Prediction rule: $y = \text{sign}(\mathbf{w}^T \mathbf{x} + b)$

Given: Training data $\{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_N, y_N)\}$

Goal: Learn \mathbf{w} and b that achieve the **maximum margin**

Characterizing the margin

Let's assume the entire training data is correctly classified by (\mathbf{w}, b) that achieve the maximum margin

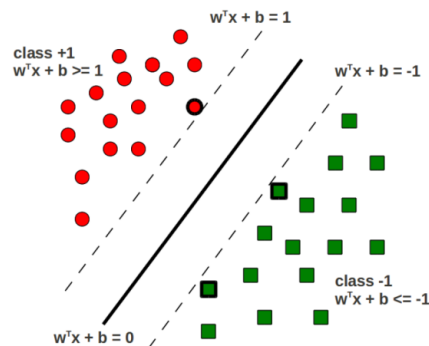


- Assume the hyperplane is such that
 - $\mathbf{w}^T \mathbf{x}_n + b \geq 1$ for $y_n = +1$
 - $\mathbf{w}^T \mathbf{x}_n + b \leq -1$ for $y_n = -1$
 - Equivalently, $y_n(\mathbf{w}^T \mathbf{x}_n + b) \geq 1$
 $\Rightarrow \min_{1 \leq n \leq N} |\mathbf{w}^T \mathbf{x}_n + b| = 1$
 - The hyperplane's margin:

$$\gamma = \min_{1 \leq n \leq N} \frac{|\mathbf{w}^T \mathbf{x}_n + b|}{\|\mathbf{w}\|} = \frac{1}{\|\mathbf{w}\|}$$

The Optimization Problem

We want to maximize the margin $\gamma = \frac{1}{\|\mathbf{w}\|}$



Maximizing the margin $\gamma = \text{minimizing } \|\mathbf{w}\|$ (the norm)

Our optimization problem would be:

$$\begin{aligned} \text{Minimize } f(\mathbf{w}, b) &= \frac{\|\mathbf{w}\|^2}{2} \\ \text{subject to } y_n(\mathbf{w}^T \mathbf{x}_n + b) &\geq 1, \quad n = 1, \dots, N \end{aligned}$$

Large Margin = Good Generalization

- Intuitively, large margins mean good generalization
 - Large margin \Rightarrow small $\|\mathbf{w}\|$
 - small $\|\mathbf{w}\| \Rightarrow$ regularized/simple solutions
- (Learning theory gives a more formal justification)

Solving the SVM Optimization Problem

Our optimization problem is:

$$\begin{aligned} \text{Minimize } f(\mathbf{w}, b) &= \frac{\|\mathbf{w}\|^2}{2} \\ \text{subject to } 1 &\leq y_n(\mathbf{w}^T \mathbf{x}_n + b), \quad n = 1, \dots, N \end{aligned}$$

Introducing **Lagrange Multipliers** α_n ($n = \{1, \dots, N\}$), one for each constraint, leads to the **Lagrangian**:

$$\begin{aligned} \text{Minimize } L(\mathbf{w}, b, \alpha) &= \frac{\|\mathbf{w}\|^2}{2} + \sum_{n=1}^N \alpha_n \{1 - y_n(\mathbf{w}^T \mathbf{x}_n + b)\} \\ \text{subject to } \alpha_n &\geq 0; \quad n = 1, \dots, N \end{aligned}$$

Solving the SVM Optimization Problem

Take (partial) derivatives of L_P w.r.t. \mathbf{w} , b and set them to zero

$$\frac{\partial L_P}{\partial \mathbf{w}} = 0 \Rightarrow \mathbf{w} = \sum_{n=1}^N \alpha_n y_n \mathbf{x}_n, \quad \frac{\partial L_P}{\partial b} = 0 \Rightarrow \sum_{n=1}^N \alpha_n y_n = 0$$

Substituting these in the **Primal** Lagrangian L_P gives the **Dual** Lagrangian

$$\begin{aligned} \text{Maximize } L_D(\mathbf{w}, b, \alpha) &= \sum_{n=1}^N \alpha_n - \frac{1}{2} \sum_{m,n=1}^N \alpha_m \alpha_n y_m y_n (\mathbf{x}_m^T \mathbf{x}_n) \\ \text{subject to } \sum_{n=1}^N \alpha_n y_n &= 0, \quad \alpha_n \geq 0; \quad n = 1, \dots, N \end{aligned}$$

Solving the SVM Optimization Problem

Take (partial) derivatives of L_P w.r.t. \mathbf{w} , b and set them to zero

A Quadratic Program for which many off-the-shelf solvers exist

$$= \sum_{n=1}^N \alpha_n y_n \mathbf{x}_n, \quad \frac{\partial L_P}{\partial b} = 0 \Rightarrow \sum_{n=1}^N \alpha_n y_n = 0$$

Substituting these into the **Primal** Lagrangian L_P gives the **Dual** Lagrangian

$$\text{Maximize } L_D(\mathbf{w}, b, \alpha) = \sum_{n=1}^N \alpha_n - \frac{1}{2} \sum_{m,n=1}^N \alpha_m \alpha_n y_m y_n (\mathbf{x}_m^T \mathbf{x}_n)$$

$$\text{subject to } \sum_{n=1}^N \alpha_n y_n = 0, \quad \alpha_n \geq 0; \quad n = 1, \dots, N$$

SVM: the solution!

Once we have the α_n 's, \mathbf{w} and b can be computed as:

$$\mathbf{w} = \sum_{n=1}^N \alpha_n y_n \mathbf{x}_n$$

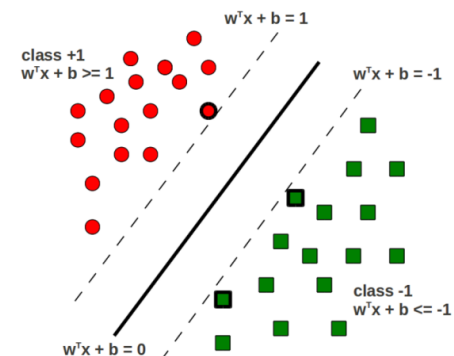
$$b = -\frac{1}{2} \left(\min_{n:y_n=+1} \mathbf{w}^T \mathbf{x}_n + \max_{n:y_n=-1} \mathbf{w}^T \mathbf{x}_n \right)$$

Note: Most α_n 's in the solution are zero (**sparse solution**)

- Reason: **Karush-Kuhn-Tucker (KKT) conditions**
- For the optimal α_n 's

$$\alpha_n \{1 - y_n(\mathbf{w}^T \mathbf{x}_n + b)\} = 0$$

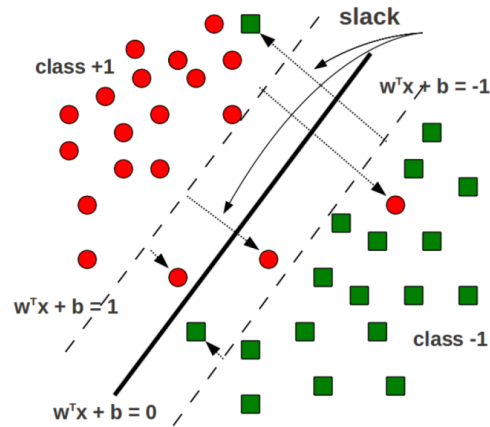
- α_n is **non-zero** only if \mathbf{x}_n lies on one of the two **margin boundaries**, i.e., for which $y_n(\mathbf{w}^T \mathbf{x}_n + b) = 1$
- These examples are called **support vectors**
- Support vectors “support” the margin boundaries



SVM in the non-separable case

- no hyperplane can separate the classes perfectly
- We still want to find the max margin hyperplane, but
 - We will allow some training examples to be **misclassified**
 - We will allow some training examples to fall **within** the margin region

SVM in the non-separable case



Recall: For the separable case (training loss = 0), the constraints were:

$$y_n(\mathbf{w}^T \mathbf{x}_n + b) \geq 1 \quad \forall n$$

For the non-separable case, we **relax** the above constraints as:

$$y_n(\mathbf{w}^T \mathbf{x}_n + b) \geq 1 - \xi_n \quad \forall n$$

ξ_n is called **slack variable** (distance \mathbf{x}_n goes past the margin boundary)

$\xi_n \geq 0, \forall n$, **misclassification when $\xi_n > 1$**

SVM Optimization Problem

Non-separable case: We will allow misclassified training examples

- .. but we want their number to be minimized
⇒ by *minimizing* the sum of slack variables ($\sum_{n=1}^N \xi_n$)

The optimization problem for the non-separable case


$$\begin{aligned} \text{Minimize } f(\mathbf{w}, b) &= \frac{\|\mathbf{w}\|^2}{2} + C \sum_{n=1}^N \xi_n \\ \text{subject to } y_n(\mathbf{w}^T \mathbf{x}_n + b) &\geq 1 - \xi_n, \quad \xi_n \geq 0 \quad n = 1, \dots, N \end{aligned}$$

C hyperparameter dictates which term dominates the minimization

- Small C => prefer large margins and allows more misclassified examples
- Large C => prefer small number of misclassified examples, but at the expense of a small margin

Soft SVM

- Same optimization as :

$$\min_{\mathbf{w}, b} \frac{\|\mathbf{w}\|^2}{2} + C \sum_{n=1}^N \max \{1 - y_n(\mathbf{w}^t \mathbf{x}_n), 0\}$$


Hinge loss!

- Why?
- Have you seen this loss function before?