These are examples of proofs used in cmsc250. These proofs tend to be very detailed. You can be a little looser.

**General Comments**

**Proofs by Mathematical Induction**

- If a proof is by Weak Induction the Induction Hypothesis must reflect that. I.e., you may NOT write the Strong Induction Hypothesis.
- The Inductive Step MUST explicitly state where the Inductive Hypothesis is used. (Something like “by IH” is good.)
Example Proof by Weak Induction

**Theorem.** For \( n \geq 1 \), \( \sum_{i=1}^{n} 4i - 2 = 2n^2 \).

**BASE CASE:** Let \( n = 1 \). The summation gives

\[
\sum_{i=1}^{n} 4i - 2 = \sum_{i=1}^{1} 4i - 2 = 4 \cdot 1 - 2 = 2 .
\]

The formula gives

\[
2n^2 = 2 \cdot 1^2 = 2 .
\]

The two values are the same.

- **INDUCTIVE HYPOTHESIS** [Choice I: From \( n - 1 \) to \( n \):]
  Assume that the theorem holds for \( n - 1 \) (for arbitrary \( n > 1 \)). Then

\[
\sum_{i=1}^{n-1} 4i - 2 = 2(n-1)^2 .
\]

[It is optional to simplify the right side. If not, it will have to be done inside the Induction Step.]

- **INDUCTIVE STEP:** [Choice Ia: Start with the sum we care about.]

\[
\sum_{i=1}^{n} 4i - 2 = \sum_{i=1}^{n-1} 4i - 2 + (4n - 2)
\]

\[
= 2(n-1)^2 + (4n - 2) \quad \text{by IH}
\]

\[
= 2(n^2 - 2n + 1) + (4n - 2) \quad \text{by algebra}
\]

\[
= 2n^2 . \quad \text{by algebra}
\]

So the theorem holds for \( n \).

- **INDUCTIVE STEP:** [Choice Ib: Start with the induction hypothesis.]

\[
\sum_{i=1}^{n-1} 4i - 2 = 2(n-1)^2
\]

\[
\sum_{i=1}^{n} 4i - 2 + (4n - 2) = 2(n-1)^2 + (4n - 2) \quad \text{adding } 4n - 2 \text{ to both sides}
\]

\[
\sum_{i=1}^{n} 4i - 2 = 2(n^2 - 2n + 1) + (4n - 2) \quad \text{merging the sum on left side}
\]

\[
\sum_{i=1}^{n} 4i - 2 = 2n^2 . \quad \text{... and algebra on the right side}
\]

\[
\sum_{i=1}^{n} 4i - 2 = 2n^2 . \quad \text{by algebra on the right side}
\]

So the theorem holds for \( n \).

- **INDUCTIVE HYPOTHESIS:** [Choice II: From \( n \) to \( n + 1 \)]

Assume that the theorem holds for arbitrary \( n \geq 1 \). Then

\[
\sum_{i=1}^{n} 4i - 2 = 2n^2 .
\]
NEED TO SHOW:  
\[ \sum_{i=1}^{n+1} 4i - 2 = 2(n+1)^2 = 2(n^2 + 2n + 1) = 2n^2 + 4n + 2. \]

- **INDUCTIVE STEP:** [Choice IIa: Start with the sum we care about.]
\[
\sum_{i=1}^{n+1} 4i - 2 = \sum_{i=1}^{n} i + (4(n+1) - 2) \quad \text{by splitting sum}
\]
\[
= 2n^2 + (4(n + 1) - 2) \quad \text{by IH}
\]
\[
= 2n^2 + (4n + 2) \quad \text{by algebra}
\]
\[
= 2n^2 + 4n + 2. \quad \text{by algebra}
\]

This is what we needed to prove, so the theorem holds for \( n + 1 \).

- **INDUCTIVE STEP:** [Choice IIb: Start with the induction hypothesis.]
\[
\sum_{i=1}^{n+1} 4i - 2 = \sum_{i=1}^{n} 4i - 2 + (4(n + 1) - 2) \quad \text{adding } 4(n + 1) - 2 \text{ to both sides}
\]
\[
\sum_{i=1}^{n+1} 4i - 2 = 2n^2 + 4n + 2 \quad \text{merging sum on the left side}
\]
\[
\quad \text{... and algebra on the right side}
\]

This is what we needed to prove, so the theorem holds for \( n + 1 \).

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**Example Proof by Strong Induction**

**BASE CASE:** [Same as for Weak Induction.]

- **INDUCTIVE HYPOTHESIS:** [Choice I: Assume true for less than \( n \)]
  (Assume that for arbitrary \( n > 1 \), the theorem holds for all \( k \) such that \( 1 \leq k \leq n - 1 \).)
  Assume that for arbitrary \( n > 1 \), for all \( k \) such that \( 1 \leq k \leq n - 1 \) that
  \[
  \sum_{i=1}^{k} 4i - 2 = 2k^2.
  \]

- **INDUCTIVE HYPOTHESIS:** [Choice II: Assume true for less than \( n + 1 \)]
  (Assume that for arbitrary \( n \geq 1 \) the theorem holds for all \( k \) such that \( 1 \leq k \leq n \).)
  Assume that for arbitrary \( n > 1 \), for all \( k \) such that \( 1 \leq k \leq n \) that
  \[
  \sum_{i=1}^{k} 4i - 2 = 2k^2.
  \]

**INDUCTIVE STEP:** [And now a brilliant proof that somehow uses strong induction.]
Constructive Induction

[We do this proof only one way, but any of the styles is fine.]

Guess that the answer is quadratic, so it has form $an^2 + bn + c$. We will derive the constants $a, b, c$ while proving it by Mathematical Induction.

**BASE CASE:** Let $n = 1$. The summation gives

$$
\sum_{i=1}^{n} 4i - 2 = \sum_{i=1}^{1} 4i - 2 = 4 \cdot 1 - 2 = 2 .
$$

The formula gives

$$
an^2 + bn + c = a \cdot 1^2 + b \cdot 1 + c = a + b + c .
$$

So, we need $a + b + c = 2$.

**INDUCTIVE HYPOTHESIS:**
Assume that the theorem holds for $n - 1$ (for arbitrary $n > 1$). Then

$$
\sum_{i=1}^{n-1} 4i - 2 = a(n-1)^2 + b(n-1) + c .
$$

[Again, it is optional to simplify the right side.]

**INDUCTIVE STEP:**

$$
\sum_{i=1}^{n} 4i - 2 = \sum_{i=1}^{n-1} 4i - 2 + (4n - 2) \quad \text{by splitting sum}
= a(n-1)^2 + b(n-1) + c + (4n - 2) \quad \text{by IH}
= a(n^2 - 2n + 1) + b(n-1) + c + (4n - 2) \quad \text{by algebra}
= an^2 + (-2a + b + 4)n + a - b + c - 2 \quad \text{by algebra}
= an^2 + bn + c . \quad \text{to make the induction work}
$$

The coefficients on each of the powers have to match. This leads to the three simultaneous equations:

$$
b = -2a + b + 4 \\
c = a - b + c - 2 \\
2 = a + b + c \quad \text{from the base case}
$$

The first equation gives $a = 2$, then the second gives $b = 0$, and finally the third gives $c = 0$. 
Constructive Induction (Another Example)

Problem: Find an upper bound on $F_n$ in the recurrence

$$F_n = F_{n-1} + F_{n-2}$$

where $F_0 = F_1 = 1$.

Guess that the answer is exponential, so $F_n \leq ab^n$. We will derive the constants $a, b$ while proving it by Mathematical Induction.

**BASE CASES:** Let $n = 0$. By definition

$$F_n = F_0 = 1$$

The formula gives

$$F_n \leq ab^n = ab^0 = a$$

So, $a \geq 1$.

Let $n = 1$. By definition

$$F_n = F_1 = 1$$

The formula gives

$$F_n \leq ab^n = ab^1 = ab$$

So, $ab \geq 1$.

**INDUCTIVE HYPOTHESIS:**
Assume that for arbitrary $n > 1$, for all $k$ such that $1 \leq k \leq n - 1$ that $F_k \leq ab^k$.

**INDUCTIVE STEP:**

$$F_n = F_{n-1} + F_{n-2} \leq ab^{n-1} + ab^{n-2} \leq ab^n.$$ 

Thus we need to solve

$$ab^{n-1} + ab^{n-2} \leq ab^n.$$ 

or

$$b^2 - b - 1 \geq 0.$$ 

By the quadratic formulas, we get

$$b \geq \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}.$$ 

Only the positive value can hold. Also, we would like the smallest possible value for $b$. So we choose

$$b = \frac{1 + \sqrt{5}}{2}.$$ 

From the base cases we get $a \geq 1$ (since the other condition is weaker), and now we would like the smallest possible value for $a$. So we choose $a = 1$. This gives

$$F_n \leq \left(\frac{1 + \sqrt{5}}{2}\right)^n.$$
Catalan Numbers

**Theorem.** For \( n \geq 1 \), \( \frac{(2n)!}{n!(n+1)!} \geq \frac{4^n}{(n+1)^2} \).

**Proof.** by Mathematical Induction.

**BASE CASE:** Easy.

**INDUCTION HYPOTHESIS:** Assume true for \( n - 1 \):

\[
\frac{(2(n-1))!}{(n-1)!n!} \geq \frac{4^{n-1}}{n^2}.
\]

**INDUCTION STEP:** Alternative I

\[
\frac{(2n)!}{n!(n+1)!} = \frac{(2n)(2n-1)}{(n-1)n} \frac{(2n-1)!}{(n-1)!n!} \\
\geq \frac{(2n)(2n-1)}{n(n+1)} \frac{4^{n-1}}{n^2} \text{ by IH} \\
= \frac{(2n)(2n-1)}{n(n+1)} \frac{(n+1)^2}{4n^2} \frac{4n^2}{(n+1)^2} \frac{4^{n-1}}{n^2} \\
= \frac{(2n)(2n-1)}{n(n+1)} \frac{4n}{4n} \\
= \frac{1}{1 - 1/n} \frac{4n}{(n+1)^2} \\
\geq \frac{4^n}{(n+1)^2}.
\]

**INDUCTION STEP:** Alternative II

\[
\frac{4^n}{(n+1)^2} = \frac{4n^2}{(n+1)^2} \frac{4^{n-1}}{n^2} \\
\leq \frac{4n^2}{(n+1)^2} \frac{(2(n-1))!}{(n-1)!n!} \text{ by IH} \\
= \frac{4n^2}{(n+1)^2} \frac{n(n+1)}{(2n)(2n-1)} \frac{(2n)(2n-1)}{n(n+1)} \frac{(2n)!}{(n-1)!n!} \\
= \frac{1}{1 + 1/(2n)} \frac{(2n)!}{n!(n+1)!} \\
\leq \frac{2n!}{n!(n+1)!}.
\]

\( \square \)