CMSC 351
Introduction to Probability Theory*

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*: Some slides are adopted from slides by Rong Jin

Outline

- Basics of probability theory
- Random variable and distributions: Expectation and Variance

Definition of Probability

- ***Experiment***: toss a coin twice
- ***Sample space***: possible outcomes of an experiment
  - S = {HH, HT, TH, TT}
- ***Event***: a subset of possible outcomes
  - A={HH}, B={HT, TH}
- ***Probability of an event*** : an number assigned to an event Pr(A)
  - Axiom 1: 0<= Pr(A) <= 1
  - Axiom 2: Pr(S) = 1, Pr(Ø)= 0
  - Axiom 3: For two events A and B, Pr(A∪B)= Pr(A)+Pr(B)-Pr(A∩B)
  - Proposition 1: Pr(~A)= 1- Pr(A)
  - Proposition 2: For every sequence of disjoint events $\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i)$

Joint Probability

- For events A and B, **joint probability** Pr(AB) (also shown as Pr(A ∩ B)) stands for the probability that both events happen.
- Example: A={HH}, B={HT, TH}, what is the joint probability Pr(AB)?

Zero

Independence

- Two events ***A and B are independent*** in case

$$\Pr(AB) = \Pr(A)\Pr(B)$$

- A set of events $\{A_i\}$ is ***independent*** in case

$$\Pr\left(\bigcap_i A_i\right) = \prod_i \Pr(A_i)$$

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- Example: Drug test

	Women	Men
Success	200	1800
Failure	1800	200

A = {A patient is a Woman}

B = {Drug fails}

Will event A be independent from event B ?

Pr(A)=0.5, Pr(B)=0.5, Pr(AB)=9/20

Independence

- Consider the experiment of tossing a coin twice
- Example I:
  - A = {HT, HH}, B = {HT}
  - Will event A independent from event B?
- Example II:
  - A = {HT}, B = {TH}
  - Will event A independent from event B?
- Disjoint ≠ Independence
- If A is independent from B, B is independent from C, will A be independent from C?

Not necessarily, say A=C

Conditioning

- If A and B are events with Pr(A) > 0, the ***conditional probability of B given A*** is

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)}$$

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Pr(B|A) = ?

Pr(A|B) = ?

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- Example: Drug test

	Women	Men
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A = {Patient is a Woman}

B = {Drug fails}

Pr(B|A) = 18/20

Pr(A|B) = 18/20

- Given A is independent from B, what is the relationship between Pr(A|B) and Pr(A)?

Pr(A|B)= P(A)

Outline

- Basics of probability theory
- Bayes' rule
- Random variable and probability distribution: Expectation and Variance

Random Variable and Distribution

- A ***random variable X*** is a numerical outcome of a random experiment
- The ***distribution*** of a random variable is the collection of possible outcomes along with their probabilities:
  - Discrete case: $\Pr(X = x) = p_\theta(x)$
  - Continuous case: $\Pr(a \le X \le b) = \int_a^b p_\theta(x)dx$
- The ***support*** of a discrete distribution is the set of all x for which $\Pr(X=x) > 0$
- The ***joint distribution*** of two random variables X and Y is the collection of possible outcomes along with the joint probability $\Pr(X=x, Y=y)$.

Random Variable: Example

- Let S be the set of all sequences of three rolls of a die. Let X be the sum of the number of dots on the three rolls.
- What are the possible values for X?
- Pr(X = 3) = 1/6*1/6*1/6=1/216,
- Pr(X = 5) = ?

Expectation

- A random variable X~Pr(X=x). Then, its expectation is

$$E[X] = \sum_x x \Pr(X = x)$$

  - In an empirical sample, x_1, x_2,…, x_N,

$$E[X] = \frac{1}{N}\sum_{i=1}^{N} x_i$$

- Continuous case: $E[X] = \int_{-\infty}^{\infty} x p_\theta(x) dx$
- In the discrete case, expectation is indeed the average of numbers in the support weighted by their probabilities
- Expectation of sum of random variables

$$E[X_1 + X_2] = E[X_1] + E[X_2]$$

Expectation: Example

- Let S be the set of all sequence of three rolls of a die. Let X be the sum of the number of dots on the three rolls.
- Exercise: What is E(X)?

- Let S be the set of all sequence of three rolls of a die. Let X be the product of the number of dots on the three rolls.
- Exercise: What is E(X)?

Variance

- The variance of a random variable X is the expectation of $(X-E[X])^2$:

$$Var(X)=E[(X-E[X])^2]$$

$$=E[X^2+E[X]^2-2XE[X]]=$$

$$=E[X^2]+E[X]^2-2E[X]E[X]$$

$$=E[X^2]-E[X]^2$$