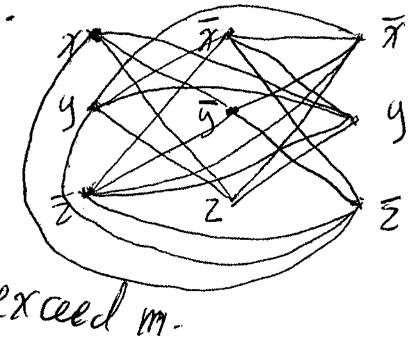
NP-completeness:

<u>clique</u> C of a graph G is a subgraph of G such that all vertices in C are connected to each other

<u>The problem</u>: Given an undirected graph $G=(V,E)$ and an integer k, determine whether G contains a clique of size $\geq k$.

Thm: The clique problem is NP-complete.

<u>Pf</u>: clearly the problem is NP, since given $\underbrace{\text{a certifice}}_{\text{Solution C as a}}$, we can check whether it is a clique in $O(|C|^2)$. Now we reduce 3-SAT to Clique. $\xrightarrow{\text{input}} E = E_1 \cdot E_2 \cdots E_m$ For each clause $E_i = (x+y+z)$ of 3-SAT we associate a "column" of three variables (even if they appear in other clauses). For edges, vertices from the same column are not connected. vertices from different columns are connected except when they are corresponding to the same variable appearing in complementary form. We set $k=m$ (we are free to choose k). E.g. if $S=(x+y+\bar{z})(\bar{x}+\bar{y}+z)(\bar{x}+y+\bar{z})$.



Now, we claim that G has a clique of size $\geq m$ iff E is satisfiable. Note that since we can choose at most one vertex from each column the clique size can not exceed m.

Assume E is satisfiable. Then there is a truth assignment in which each clause contains at least one true variables. Choose that varible and we get a clique of size m by construction.

The reverse, if we have a clique of size m, then since from each column, we can pick only one vertex, if we assign the corresponding variable a value 1, and other variables arbitrary, we are done. Since all vertices in the clique are connected to one another, and we made sure that x and $\bar{x}$ are never connected, this truth assignment is consistent.

Vertex Cover:

let G be a given graph. A vertex cover of G is a set of vertices such that every edge in G is incident to at least one of these vertices.

The problem: Given an undirected graph $G=(V,E)$ and an integer k, determine whether G has a vertex cover containing $\leq k$ vertices.

Thm: The vertex cover problem is NP-complete.

Pf: The vertex cover clearly belongs to NP, since given a solution of size $\leq k$ as a certificate, we can verify it easily in $O(|E|+|V|)$. Now we reduce clique to vertex cover. We have to transform an arbitrary instance of the clique into an instance of vertex cover such that the answer to the clique problem is positive iff the answer to the corresponding vertex cover problem is positive. let $G=(V,E)$ be the graph of clique. let $\bar{G}=(V,\bar{E})$ be the complement graph of G; namely $\bar{G}$ has the same set of vertices and two vertices are connected in $\bar{G}$ iff they are not connected in G. We claim that the clique problem is reduced to vertex cover represented by the graph $\bar{G}$ and $n-k$, where $n=|V|$. suppose C is a clique in G. Then the set of vertices $V-C$ covers all the edges of $\bar{G}$, because in $\bar{G}$ there are no edges connecting vertices in C (they are all in G). Thus $V-C$ is a vertex cover of size $n-k$ $\bar{G}$. Conversely let D be a vertex cover in $\bar{G}$. Then D covers all the edges in $\bar{G}$, so in $\bar{G}$ there could be no edges connecting vertices in $V-D$. Thus $V-D$ generates a clique in G. Therefore if there is a vertex cover of size k in $\bar{G}$, then there is a clique of size $n-k$ in G. The reduction is clearly poly-time since it requeres only the construction of G' from G (say in $O(|V|^2)$).

More NP-Complete problems:

- Bin Packing and Knapsack (subset sum) problems that we have seen earlier in the class.
- coloring or 3-coloring of Graphs
- Independent set: given a graph G and an integer k, whether G contains an independent set of size $\geq k$ (straightforward reduction from clique).
- Dominating set: given a graph G and an integer k, whether G contains a dominating set of size at most k (i.e. a set D of vertices in G such that every vertex of G is either in D or is adjacent to at least one vertex from D). Reduction from vertex cover.
- Hamiltonian cycle: A Hamiltonian cycle in a graph is a simple cycle that contains each vertex exactly once. The problem is to determine whether a given graph contains a Hamiltonian circuit. The problem is NP-complete for both undirected and directed graphs by reduction from vertex cover.
- Hamiltonian path: the same as Hamiltonian cycle, but we want a simple path instead of a simple cycle (reduction from vertex cover).
- Set cover: Given a set of elements $\{1, 2, \ldots, m\}$ (called the universe) and n sets whose union comprises the universe, and an integer k, the set cover problem is find at most k sets whose union still contains all elements of the univers (reduction from vertex cover).

Set-cover esp. is very important problem in practical applications.

Note that in general for all NP-complete problems we can use backtracking and branch and bound techniques to obtain the optimum solution (in exponential time). Alternatively, we can use approximation algorithms to obtain an approximate solution in poly-time.