CMSC 351 - Introduction to Algorithms
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Lecture 1

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1 Introduction

Welcome to CMSC 351 (Introduction of Algorithms). The full syllabus is available on the course website. The prerequisites for the course (see the course catalog) are CMSC 250 $\leftarrow$ MATH 141 and CMSC 212 $\leftarrow$ CMSC 132. We have homeworks, 2 midterms and a final exam. There will be a **bonus** programming project in addition to **bonus** of 5% in the homeworks. Information about office-hours and email contacts of the TAs and the instructor are available on the course website. Read the full syllabus carefully.

In this class you learn what is an **algorithm**? You will learn some types of algorithms (and not all; you may take CMSC 451 to have a better understanding and a more complete picture). We will also learn about the running times of algorithms in this class.

The word "algorithm" comes from the name of Al-Khwarizmi, Muhammad (a Persian scientist, mathematician and author) who developed the concept of an algorithm in mathematics. An algorithm is an effective method for solving a problem expressed as a finite sequence of steps. Each algorithm is a list of well-defined tasks instructions for computing a task. Starting from an initial state, the instructions describe a computation that proceeds through a well-defined series of successive states, eventually terminating in a final ending state.

2 Mathematical Induction

As you have learned in CMSC 250, mathematical induction is a very powerful proof technique that plays a major role in algorithm design. In this session we briefly review induction through some examples.

It usually works as follows. Let T be a theorem (statement that we want to prove). Suppose T includes a parameter n whose value can be any natural number (a positive integer). Instead of proving directly that T holds for all values of n, we prove the following two conditions:

1. **Basis of the induction**: T holds for $n = 1$
2. **Inductive Hypothesis**: For every $n > 1$, if T holds for $n - 1$, then T holds for n.

The reason that these two conditions are sufficient is clear. Conditions 1 and 2 imply directly that T holds for $n = 2$. If T holds for $n = 2$, then condition 2 implies that T holds for $n = 3$ and so on. The induction principle itself is so basic that we consider it as an axiom and it is usually not proved. Proving induction hypothesis is easier in many cases than proving the theorem directly since it comes for free.

Thus the induction principle is defined as follows:

Induction Principle
If a statement P, with a parameter n, is true for $n = 1$, and if for every $n > 1$, the truth of P for $n - 1$ implies its truth for n, then P is true for all natural numbers.

Remarks:

- Sometimes we use n and $n + 1$ instead of $n - 1$ and n.
- It is called **Strong Induction** if we use the truth of P for all natural numbers less than n instead of just the truth of P for $n - 1$.

3 Three simple examples of induction

More examples and exercises on induction are available in Chapter 2 of the book [1] for this course.

3.1 Example 1

Theorem 1 *For all natural numbers* x *and* n*,* $x^n - 1$ *is divisible by* $x - 1$*.*

Proof: The proof is by induction on n. The theorem is trivially true for $n = 1$. We assume $x^{n-1} - 1$ is divisible by $x - 1$ for all natural numbers x. We now prove that $x^n - 1$ is divisible by $x - 1$. The idea is to try to write the expression $x^n - 1$ using $x^{n-1} - 1$, i.e., $x^n - 1 = x(x^{n-1} - 1) + (x - 1)$. The second term is clearly divisible by $x - 1$ while the first term is divisible by $x - 1$ due to the induction hypothesis. ■

3.2 Example 2

Theorem 2 *The sum of the first* n *natural number is* $\frac{n(n+1)}{2}$.

Proof: The proof is by induction on n. For $n = 1$ we have $1 = \frac{1 \times 2)}{2} = 1$. We assume the sum of the first n natural numbers is $S(n) = \frac{n(n+1)}{2}$. Then $S(n+1) = S(n) + (n+1)$. By induction hypothesis $S(n) = \frac{n(n+1)}{2}$. So $S(n+1) = \frac{n(n+1)}{2} + (n + 1) = \frac{(n+1)(n+2)}{2}$, which is exactly what we wanted to prove. ■

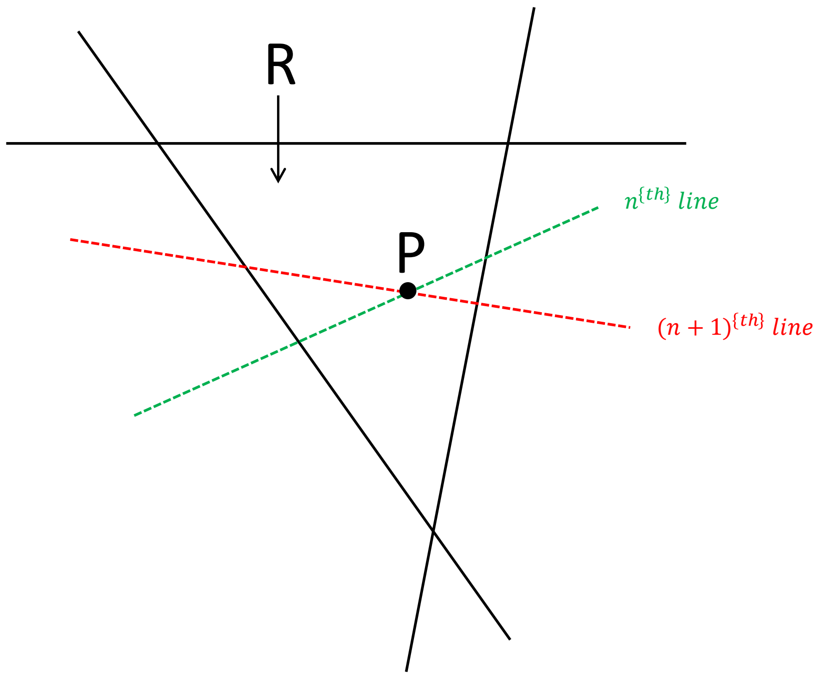

Figure 1: Lines in general position

3.3 Example 3

Theorem 3 *If n is a natural number and $1 + x \geq 0$ then $(1 + x)^n \geq 1 + nx$.*

Proof: The proof is by induction on n. For $n = 1$ both sides are equal to $1+x$. By induction hypothesis $(1+x)^n \geq 1+nx$. Then $(1+x)^{n+1} = (1+x)^n(1+x)$. Now induction hypothesis implies $(1 + x)^n \geq 1 + nx$. Also $1 + x \geq 0$. Hence $(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1+(n+1)x+nx^2 \geq 1+(n+1)x$ as $nx^2 \geq 0$. ■

4 More deeper examples

Now we look at more involved examples of induction.

4.1 Example 4

Definition 1 *A set of lines in the plane are said to be in **general position** if no two lines are parallel and no three lines intersect at a common point.*

Theorem 4 *The number of regions* $P(n)$ *in the plane formed by* n *lines in general position is* $\frac{n(n+1)}{2} + 1$.

Proof: First let us see how we obtain this number $\frac{n(n+1)}{2} + 1$. A good hint for the right guess can be obtained from small examples: for $n = 1$, there are 2; for $n = 2$, there are 4 and for $n = 3$, there are 7. It seems that adding one more line to $n-1$ lines in general position in the plane increases the number of regions by n. If this guess is correct then the rest of the proof is similar to the $S(n)$ we have seen in Example 2. For $n = 1$ we have $P(1) = 2$. By the guess $P(n+1) = P(n) + (n+1) = 1 + \frac{n(n+1)}{2} + (n+1) = 1 + \frac{(n+1)(n+2)}{2}$.

We can also prove the guess by induction. For $n = 1$, adding one line adds one region (from one region to two). Refer to Figure 1 and see the book [1] for more details. Let us remove the n^{th} line for now. Then we have n lines. Thus we have n new regions by adding the $(n+1)^{th}$ line to the set of lines $1, 2, \ldots, n-1$. Now let us pull the n^{th} line back. Since all lines are in general position, the n^{th} and the $(n+1)^{th}$ lines intersect at a point P which must be inside a region R. Both lines thus intersect R. So the addition of the $(n+1)^{th}$ line, when the n^{th} line is not present cuts R into 2 while when the n^{th} line is present affects R by adding two more regions instead of just adding one region. Furthermore R is the only region affected so far, i.e., in all regions with presence of the n^{th} line and the $(n+1)^{th}$ line we have added exactly one region. Any two lines in general position meet at only one point. Hence the $(n+1)^{th}$ line adds n regions without the presence of the n^{th} line but it adds $n+1$ regions with the presence of the n^{th} line which completes the proof. ■

Note that we have used **two inductions** here: one inside the other.

4.2 Example 5

Definition 2 *Consider* n *distinct lines in the plane not necessarily in general position. Two regions are called as* ***neighboring*** *if and only if they have an edge in common. An assignment of colors to the regions formed by these lines is called as a* ***valid coloring*** *if neighboring regions have different colors.*

Theorem 5 *It is possible to have a valid coloring of any number of lines in the plane with only two colors.*

Proof: For $n = 1$ it is trivial. Assume it is correct for $n-1$. Consider the n^{th} line. The only question is how to modify the coloring when the n^{th} line is added. Consider the following algorithm:

1. Divide the regions into two groups according to which side of the n^{th} line they lie.
2. Leave all regions on one side colored the same as before, and reverse the colors of all regions on the other side.

To prove that the algorithm works consider two neighboring regions R_1 and R_2 and the edge common between them. If both are on the same side of the n^{th} line, then by the induction hypothesis they were colored differently before the n^{th} line was added and so even if both colors get reversed still they will remain different. If the edge between them is part of the n^{th} line, then they belong to the same region before the n^{th} line was added. Since the color of one region was reversed, they are now colored differently. ∎

See more examples of induction in the book [1].

References

[1] Udi Manber, *Introduction to Algorithms - A Creative Approach*